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# Analytic approach on maximum Bayesian entropy ordered weighted averaging operators

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ABSTRACT

In this paper the maximum Bayesian entropy model for obtaining the ordered weighted averaging (OWA) operators was analyzed. The model is based on a given prior OWA vector and the obtained weights are called the maximum Bayesian entropy OWA (MBEOWA) weights. An analytic form for the MBEOWA weights was determined by using the Lagrange multipliers method and some properties of this form were investigated. Finally, a general model for determining the OWA weights based on the Bayesian entropy was introduced and discussed which included some other models. The efficiency of the proposed model was demonstrated by some examples.



## 1. Introduction

The ordered weighted averaging (OWA) as aggregation operators was initially introduced by Yager (1988). The OWA operator has been used widely in different research fields (Emrouznejad & Marra, 2014) such as: decision making (El Bouri & Amin, 2015; Oukil & Amin, 2015; Zhanga, Dongb, & Xu, 2013), fuzzy logic (Vigiera, Schergerb, & Terceo, 2016), data mining (Cao, Yu, & Zhang, 2015; Torra, 2004) and social network analysis (Hea, Liub James, Hub, & Wang, 2015, Wanga et al., 2015).

The important issue on the theory of the OWA operator weights was the determination of the associated weights. Many models were suggested by different researchers to obtain the optimal OWA operator weights. Yager (1988) defined the degree of orness and introduced a measure of dispersion based on the Shannon entropy for the OWA weights. O'Hagan (1988) proposed a maximum entropy model for determining the OWA operator weights based on the principle of maximum entropy and subject to specific degree of orness the weights are so-called MEOWA. Filev and Yager (1995) obtained an analytic form of the MEOWA weights and described some of their properties. Later on, Fuller and Majlender (2001) obtained the MEOWA weights via a polynomial equation. Wu, Sun, Liang, and Shan-Lin (2009) proposed a linear programming model for obtaining the OWA weights based on the maximum Yager's entropy. Yari and Chaji (2012) suggested the maximum Bayesian entropy approach based on a given prior OWA vector for obtaining the OWA weights.

In this work the Lagrange multipliers method was used to solve the maximum Bayesian entropy OWA model for determining the MBEOWA weights and then a parametric form of these weights was discussed. This parametric form gives us a direct method for obtaining the MBEOWA weights and enables us to study the weights deeper, as well as accesses us to analyze some interesting properties of the weights. At the follows, a general maximum Bayesian entropy model for obtaining the OWA weights was proposed containing some other models based on the entropy. The general model is more efficient than the related existing models for determining the OWA weights and allows us to obtain the OWA weights according to a suitable function of inputs.

## 2. Maximum Bayesian entropy model for obtaining the OWA weights

An OWA operator of dimension n is a mapping of $F_W : \mathcal{R}^n \longrightarrow \mathcal{R}$ that has an associated vector $W^T = [w_1,\ w_2, \ldots, w_n]$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. The aggregated value of $w_1,\ w_2, \ldots, w_n$ determined with the function value of $F_w(x_1, x_2, \ldots, x_n)$ is as following:

$$F_W(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} w_i y_i. \tag{1}$$

where $y_i$ is the $i$th largest element of the arguments. A measure is called the *orness measure* for characterizing the degree to which the aggregation is max-like or min-like operation defined by Yager (1988) as follows:

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$$\alpha = \text{orness}(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \leqslant \alpha \leqslant 1.$$

The max, min and average aggregation operators are: $w^* = (1,0,\ldots,0), w_* = (0,0,\ldots,1)$ and $w_A = (\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$ respectively. Then we have $\text{orness}(w^*) = 1, \text{orness}(w_*) = 0$ and $\text{orness}(w_A) = \frac{1}{2}$, therefore we obtain: $F_{w^*}(X) = y_n, F_{w_*}(X) = y_1$ and $F_{W_A}(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$. The measure of dispersion using the Shannon entropy for $W$ as a weighting vector with elements $w_1, w_2, \ldots, w_n$ which determine the degree of information about the individual aggregates in the aggregation process are defined by Yager (1988) as below:

$$E(W) = -\sum_{i=1}^{n} w_i \ln w_i$$

O'Hagan (1988) proposed a model based on maximum entropy principle to generate the OWA operator weights called maximum entropy OWA (MEOWA) operators. The MEOWA weights were determined by the following nonlinear optimization problem:

$$Max\ disp(W) = -\sum_{i=1}^{n} w_i \ln w_i$$

$$s.t. orness\ (W) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 \leqslant \alpha \leq 1 \tag{2}$$

$$\sum_{i=1}^{n} w_i = 1, \quad w_i \in [0,1], \quad i = 1,2,\ldots,n.$$

where $\alpha$, as the specific degree of orness, is determined by a decision-maker. Kapur (1998) defined another measure of the entropy called Bayesian entropy. Yari and Chaji (2012) utilized the measure and the principle of maximum entropy and introduced the following model for generating the OWA operator weights based on a given prior OWA weight vector $W^{prior} = (\beta_1,\ \beta_2,\ \ldots,\ \beta_n)$:

$$Max(W) = -\left[\sum_{i=1}^{n} w_i \ln \frac{w_i}{\beta_i}\right] - \ln(\beta_i)_{\min}$$

$$s.t. orness(w) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \leqslant \alpha \leqslant 1 \tag{3}$$

$$\sum_{i=1}^{n} w_i = 1,\ w_i \in [0,1], \quad i = (1,2,\ldots,n).$$

The associated weights called the maximum Bayesian entropy OWA (MBEOWA) operators.

### 3. Analytic form of the maximum Bayesian entropy OWA weights

In the following, we used the Lagrange multiplier method to obtain an analytic solution for determining the MBEOWA operator weights. This will enable us to analyze the MBEOWA weights deeper and prove some properties of the weights, as well as to simplify the process used for obtaining the MBEOWA weights.

The Lagrange function of the objective function (3) subject to the constraints of the model were as $\lambda$ and $\gamma$ are real numbers is as following:

$$L = -\left[\sum_{i=1}^{n} w_i \ln \frac{w_i}{\beta_i}\right] - \ln(\beta_i)_{\min} + \lambda\left(\sum_{i=1}^{n} \frac{n-i}{n-1} w_i - \alpha\right) + \gamma\left(\sum_{i=1}^{n} w_i - 1\right).$$

Taking the partial derivatives of L with respect to $w_i, \lambda$ and $\gamma$ and setting them equal to zero the following are obtained:

$$\frac{\partial L}{\partial w_i} = -\ln \frac{w_i}{\beta_i} - 1 + \lambda\ \frac{n-i}{n-1} + \gamma = 0, \quad \text{for } i = 1,\ 2, \ldots,\ n \tag{1-4}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i - \alpha = 0, \tag{2-4}$$

$$\frac{\partial L}{\partial \gamma} = \sum_{i=1}^{n} w_i - 1 = 0. \tag{3-4}$$

From (1-4) and (2-4) we have

$$w_i = \beta_i e^{\lambda\ \frac{n-i}{n-1} + \gamma - 1}, \quad i = 1,\ 2,\ \ldots,\ n$$
$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \beta_i e^{\lambda\ \frac{n-i}{n-1} + \gamma - 1} = 1.$$

Thus,

$$w_i = \frac{\beta_i e^{\lambda\ \frac{n-i}{n-1}}}{\sum_{i=1}^{n} \beta_i e^{\lambda\ \frac{n-i}{n-1}}}, \quad i = 1,\ 2,\ \ldots,\ n. \tag{5}$$

and

$$\alpha_n(\lambda) = \text{orness}(w) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\frac{\beta_i e^{\lambda\ \frac{n-i}{n-1}}}{\sum_{i=1}^{n} \beta_i e^{\lambda\ \frac{n-i}{n-1}}}. \tag{6}$$

For $i = n$ and $i = 1$ from Eq. (1-4) we have

$$-\ln \frac{w_n}{\beta_n} + \gamma - 1 = 0 \iff \gamma = 1 + \ln \frac{w_n}{\beta_n},$$

$$-\ln \frac{w_1}{\beta_1} - 1 + \lambda + \gamma = 0 \iff -\ln \frac{w_1}{\beta_1} - 1 + \lambda + \gamma = 0.$$

Thus,

$$\lambda = \ln \frac{w_1}{\beta_1} + 1 - \gamma = \ln \frac{w_1}{\beta_1} + 1 - \left(1 + \ln \frac{w_n}{\beta_1}\right) = \ln \frac{w_1}{\beta_1} - \ln \frac{w_n}{\beta_1}.$$

Therefore, from (1-4) for any $i$ we can find

$$\ln \frac{w_i}{\beta_i} = \lambda \frac{n-i}{n-1} + \gamma - 1 = \frac{n-i}{n-1}\left(\ln \frac{w_1}{\beta_1} - \ln \frac{w_n}{\beta_n}\right) + \ln \frac{w_n}{\beta_n}.$$

Thus,

$$\ln \frac{w_i}{\beta_i} = \frac{i-1}{n-1} \ln \frac{w_n}{\beta_n} + \frac{n-i}{n-1} \ln \frac{w_1}{\beta_1}$$

$$w_i = \beta_i \sqrt[(n-1)]{\left(\frac{w_n}{\beta_n}\right)^{i-1}\left(\frac{w_1}{\beta_1}\right)^{n-i}}. \tag{7}$$

If $\frac{w_n}{\beta_n} = \frac{w_1}{\beta_1}$ then (7) gives

$$\frac{w_1}{\beta_1} = \frac{w_2}{\beta_2} = \ldots = \frac{w_n}{\beta_n}.$$

Obviously, if $\beta_i = \beta_j$ for all $i,j = 1,\ 2,\ \ldots,\ n$ then the weights are as $w_1 = \ldots = w_n = \frac{1}{n}$ which is the optimum solution of the constrain problem (3) when the $\text{orness}(w) = 0.5$ and the prior weights are equal.

**Example 1.** Assume $n = 3$ and $\beta_1 = 0.85$, $\beta_2 = 0.1$ and $\beta_3 = 0.05$, we can find from (5)

$$w_1 = \frac{0.85e^{\lambda}}{0.85e^{\lambda} + 0.1e^{\lambda/2} + 0.05}, \quad w_2 = \frac{0.1e^{\lambda/2}}{0.85e^{\lambda} + 0.1e^{\lambda/2} + 0.05},$$
$$w_3 = \frac{0.05}{0.85e^{\lambda} + 0.1e^{\lambda/2} + 0.05}.$$