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A posteriori error estimates and adaptive procedures for the meshless Galerkin boundary node method for 3D potential problems

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ABSTRACT

The Galerkin boundary node method (GBNM) is a boundary only meshless method that combines variational formulations of boundary integral equations with the moving least-squares approximations. This paper presents the mathematical derivation of a posteriori error estimates and adaptive refinement procedures for the GBNM for 3D potential problems. Two types of error estimators are developed in detail. One is a perturbation error estimator that is formulated based on the difference between numerical solutions obtained using two successive nodal arrangements. The other is a projection error estimator that is formulated based on the difference between the GBNM solution itself and its L^2-orthogonal projection. The reliability and efficiency of both types of error estimators is established. That is, these error estimators are proven to have an upper and a lower bound by the constant multiples of the exact error in the energy norm. A localization technique is introduced to accommodate the non-local property of integral operators for the needed local and computable a posteriori error indicators. Convergence analysis results of corresponding adaptive meshless procedures are also given. Numerical examples with high singularities illustrate the theoretical results and show that the proposed adaptive procedures are simple, effective and efficient.

1. Introduction

Meshless (or meshfree) methods are attractive computational techniques for numerical solutions of partial differential equations as they can alleviate the difficulty of meshing and remeshing the entire structure via simply adding or deleting nodes. Compared to conventional computational engines formulated based on mesh such as the finite element method (FEM) and the boundary element method (BEM), the prominent feature of meshless methods is the absence of an explicit mesh, and the approximate solutions are generated entirely based on a set of scattered nodes. In the past two decades, meshless methods have drawn much attention and gained great success in the field of computational science and engineering [1,2]. Many types of meshless methods have been developed so far. These methods can be classified into two categories: the domain type and the boundary type. The domain type meshless methods are represented by the element free Galerkin (EFG) method [1,2], the reproducing kernel particle method (RKPM) [1], the *h*–*p* meshless method [1], the finite point method (FPM) [3], the generalized finite element method (GFEM) [4], the point interpolation method (PIM) [2] and the smoothed finite element method (SFEM) [5]. These methods followed the idea as the FEM, in which the problem domain is discretized by nodes.

Boundary integral equations (BIEs) and BEMs have been broadly used for the numerical solution of a variety of boundary value problems as they can reduce the computational dimensions of the original problem by one and give a simple discretization of the exterior problems. The boundary type meshless methods are developed by the combination of the meshless idea with BIEs, such as the boundary node method (BNM) [6], the meshless local boundary integral equation (LBIE) method [7], the boundary point interpolation method [2], the hybrid boundary node method [8–11], the boundary element-free method [12], the boundary face method [13] and the Galerkin boundary node method (GBNM) [14,15].

Adaptive procedures, which use the currently available computed information to steer the computational process, play an increasingly decisive role in scientific computations. In the last four decades, there have appeared numerous computational reports of the adaptive FEM and the adaptive BEM [16–19]. In these methods, adaptive techniques have become important tools for increasing the reliability and reducing the cost of numerical computations. The FEM and the BEM depend on the generation of a mesh, adapted or not. In some cases, this can be time-consuming and very difficult, especially for adaptive analysis. In meshless

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methods, since no predefined nodal connectivity or mesh is used for field variable approximation, mesh-related difficulties can be avoided. This salient feature simplifies significantly the implementation for adaptive meshless schemes, as nodes can be conveniently inserted or removed for the refinement or coarsening procedures. The subject of adaptive procedures for meshless methods and a consequent adaptive analysis is crucial to the effective implementation of meshless algorithms for practical engineering computation. During the past two decades, a large amount of research has been devoted to developing adaptive algorithms based on a posteriori error estimation for domain type meshless methods such as the h–p meshless method [20], the EFG [21–24], the RKPM [25,26], the GFEM [4], the FPM [3,27], the PIM [28] and the SFEM [29]. Some significant advances have been achieved in the theory and implementation of the adaptive procedures for these meshless methods.

As the BEM, BIEs-based meshless methods have emerged as promising numerical techniques in scientific computing. However, because of the non-local character of integral operators and the essential differences between BIEs and differential equations, the current state-of-technology of adaptive analysis of such methods is barely at the introductory level. To our knowledge, the first attempt on the adaptive procedures for BIEs-based meshless methods was made by Chati et al. [17,30]. They have pioneered error indicators and adaptive algorithms for the BNM using hypersingular residual techniques similar to those used in the BEM [17]. Besides, Guo and Chen [31] have developed an adaptive algorithm for the LBIE method based on the dual error indicators. The efficiency of these adaptive algorithms has been demonstrated numerically, but the corresponding mathematical foundation is absent.

The GBNM is a boundary type meshless method that combines a variational formulation of BIEs for governing equations with the moving least-squares (MLS) approximations for generation of the trial and test functions. Unlike other MLS-based boundary type meshless methods, boundary conditions in the GBNM are incorporated into the variational formulation, thus they can be directly and easily implemented without the addition of extra equations. Another prominent feature of the GBNM is the conservation of the symmetry and positive definiteness of the variational formulation in the process of numerical implementation. This method has been successfully tried for problems in potential theory [14,15], linear elasticity [32] and fluid mechanics [33,34]. The corresponding convergence analysis and a priori error estimate have been established theoretically. These mathematical proofs guarantee that this method will converge to the true solution.

Very recently, the GBNM has been extended for a posteriori error estimate and adaptivity for 2D problems [35]. In that work the problem is solved twice by two successive nodal arrangements. The numerical solution in the second analysis is expected to be more accurate than the initial and then, an a posteriori perturbation error estimator is defined as the difference between both solutions. The theoretical proof in Ref. [35] shows that this perturbation error estimator is always efficient and is reliable under a saturation assumption. This estimator was coupled with an h-refinement technique to form an effective adaptive meshless algorithm for boundary-only analysis of boundary value problems. The results in Ref. [35] were only shown for the 2D case. One aim of this paper is to extend these results to the 3D case.

In the perturbation error estimator, the numerical solution corresponding to the initial coarse nodal arrangement has to be computed, which is a temporary result only. In principle, the side results should be computed with as less computational cost as possible. In order to avoid the direct computation of the side result, the present paper also develops a projection error estimator that is achieved basically by calculating the difference between the GBNM solution itself and its L^2-orthogonal projection. We provide a proof that the projection error estimator is equivalent to the perturbation error estimator. With the aid of the a posteriori error estimates and a localization technique, we define local error indicators and we present adaptive algorithms for h-adaptivity for 3D problems. The local error indicators allow direction control of the local refinements and can be easily computed for each individual cell. The projection error estimator is reminiscent of the Zienkiewicz–Zhu error estimator which is an often employed tool for the a posteriori finite element error analysis and which enjoys a high popularity in engineering because of its striking simplicity and universality [36]. The idea of the Zienkiewicz–Zhu estimator was adopted in some domain type meshless methods [21,23]. The proposed projection error estimator differs from traditional Zienkiewicz–Zhu-type error estimators in that it uses L^2-orthogonal projection for boundary variables instead of stress smoothing techniques for domain variables and evaluates error based on individual cells instead of elements.

For mesh-based numerical methods, the mathematical proof of convergence of some adaptive algorithms has been developed [37–39]. For meshless methods, although adaptive schemes have attracted considerable attention and perform very efficiently in practice, not much is rigorously known on the convergence. A first rigorous mathematical research on the convergence of adaptive meshless schemes was given in Ref. [35] for 2D problems. In this work, the convergence proof of the developed adaptive meshless algorithms is also established for 3D problems.

The rest of this paper is organized as follows. Section 2 gives some notations to be used later. Section 3 outlines the formulations of the GBNM for 3D problems. Sections 4 and 5 provide two adaptive procedures based on the perturbation error estimator and the projection error estimator, respectively. The convergence analysis of the developed adaptive algorithms is presented in Section 6. Numerical examples are given in Section 7. Finally, conclusions are summarized in Section 8.

2. Notations

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary Γ and let Ω' denote the complementary of $\overline{\Omega}=\Omega\cup\Gamma$, that is the exterior of $\overline{\Omega}$. A general point of $\mathbb{R}^3$ is denoted by $\mathbf{x}=(x_1,x_2,x_3)$. Assume that the bounding surface Γ is the union of piecewise smooth segments called panels. On each panel, surface curvilinear coordinates are defined by $\mathbf{s}=(s_1,s_2)$. For any point $\mathbf{x}\in\Gamma$, assume that the influence domain of $\mathbf{x}$ is a circle $\Re(\mathbf{x})$. To avoid the discontinuity at edges and corners, we restrict the domain of influence of a boundary point to the panel to which it belongs. Consequently, the MLS approximation can be carried out on each panel independently.

Let $\mathcal{T}_N$ denote a triangulation of the boundary surface Γ into N non-overlapping cells Γ_i, then $\Gamma=\bigcup_{i=1}^{N}\Gamma_i$. Besides, we use $h\in L^\infty(\Gamma)$ to denote the local cell-size function, which can be defined as

$$h|_{\Gamma_i}:=h_i:=\mathrm{diam}(\Gamma_i),\quad \Gamma_i\in\mathcal{T}_N$$

We stress that Γ_i is not a boundary element, and thus no shape function is dependent on it. In the MLS-based boundary type meshless methods, boundary nodes can be located along cells and the MLS approximations are defined by means of nodes alone. In meshless approaches, the number of nodes corresponding to each cell is arbitrary. Nevertheless, to carry out accurate integration via Gaussian quadrature, it is recommended to introduce a small number of nodes per cell [6,20,22]. It has been shown in Refs. [15,17,30] that one node per cell and the location of the node at