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Computing optimal replacement time and mean residual life in reliability shock models

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ABSTRACT

In this paper, matrix-based methods are presented to compute the optimal replacement time and mean residual lifetime of a system under particular class of reliability shock models. The times between successive shocks are assumed to have a common continuous phase-type distribution. The system's lifetime is represented as a compound random variable and some properties of phase-type distributions are utilized. Extreme shock model, run shock model, and generalized extreme shock model are shown to be the members of this class. Graphical illustrations and numerical examples are presented for the run shock model when the interarrival times between shocks follow Erlang distribution.

1. Introduction and preliminaries

Shocks models have always been very popular in applied probability and engineering reliability. In a shock model, a system (or component) is subject to shocks of random magnitudes at random times and it fails if a certain pattern corresponding to shocks, and/or times between shocks occurs. Thus the failure time of the system can be represented by a compound random variable which appears as a function of magnitudes of shocks and times between consecutive shocks. Various shock models have been defined and studied in the literature. They can be classified as cumulative shock models, extreme shock models, run shock models, delta shock models, and mixed shock models. Recently, many research papers on reliability shock models have been published in probability and engineering journals. Wen, Cui, Si, and Liu (in press) studied a multiple warm standby delta shock system. Parvardeh and Balakrishnan (2015) have obtained some results on reliability characteristics of a system under mixed delta shock models. Cha and Finkelstein (2016) presented new shock models based on the generalized Polya process. Eryilmaz (2016) studied two different discrete time shock models when the shocks occur according to a Markov chain. Zhou, Wu, Li, and Xi (2016) proposed a periodic preventive maintenance modeling method for leased equipment with continuous internal degradation and stochastic external shock damage. Song, Coit, and Feng (2016) developed new reliability models for systems subject to competing hard and soft failure processes with shocks that have dependent effects. In An and Sun (2017), a reliability model for systems subject to multiple dependent competing failure processes with shock loads above a certain level has been proposed. Mercier and Pham (2017) considered a bivariate failure time model with random shocks and mixed effects.

In this paper, we propose a method to compute optimal replacement time and mean residual lifetime of the system that is defined under a particular class of shock models. In this class, the number of shocks that cause failure of the system and the times between successive shocks are assumed have phase-type distributions. Our method is based on closure properties of phase-type distributions.

Before proceeding further we fix some notation.

N: The random variable representing the number of shocks that cause failure of the system.
PH_d: Discrete phase-type distribution.
PH_c: Continuous phase-type distribution.
X_i: The interarrival time between $(i-1)$-th and i-th shocks, $i \geqslant 1$.
F: The common cumulative distribution function of $X_1, X_2, \ldots$.
T: System's lifetime.

A discrete phase type distribution can be seen as the distribution of the time to absorption in an absorbing Markov chain. The probability mass function (pmf) of N which has a discrete phase-type distribution has the form of

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$$P\{N = n\} = \mathbf{a}\mathbf{Q}^{n-1}\mathbf{u}', \tag{1}$$

for $n \in \mathbb{N}$, where $\mathbf{Q} = (q_{ij})_{m\times m}$ is a matrix which includes the transition probabilities among the m transient states, and $\mathbf{u}' = (\mathbf{I} - \mathbf{Q})\mathbf{e}'$ is a vector which includes the transition probabilities from transient states to the absorbing state, $\mathbf{a} = (a_1, \ldots, a_m)$ with $\sum_{i=1}^{m} a_i = 1$, and $\mathbf{I}$ is the identity matrix (see, e.g. Neuts, 1981). The matrix $\mathbf{Q}$ must satisfy the condition that $\mathbf{I} - \mathbf{Q}$ is nonsingular. The survival function of N is given by

$$P\{N > n\} = \mathbf{a}\mathbf{Q}^{n}\mathbf{e}', \tag{2}$$

where $\mathbf{e} = (1, \ldots, 1)_{1\times m}$. The expected value of N can be computed from $E(N) = \mathbf{a}(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{e}'$. We shall use $N \sim PH_d(\mathbf{a}, \mathbf{Q})$ to represent that the random variable N has a discrete phase-type distribution.

The cumulative distribution function of a continuous phase-type random variable is represented as

$$F(t) = P\{X \leqslant t\} = 1 - \boldsymbol{\alpha}\exp(\mathbf{A}t)\mathbf{e}', \tag{3}$$

where the nonsingular matrix $\mathbf{A}$ of dimension $m \times m$ has negative diagonal elements, and non-negative off-diagonal elements. Furthermore, all row sums of A are non-positive. We shall use $X \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$ to represent that the random variable X has a continuous phase-type distribution of order m with a PH-generator $\mathbf{A}$ and substochastic vector $\boldsymbol{\alpha}$, i.e. $\boldsymbol{\alpha}\mathbf{e}' \leqslant 1$. If the vector $\boldsymbol{\alpha}$ is properly substochastic, i.e. $\boldsymbol{\alpha}\mathbf{e}' < 1$ then the distribution of the random variable X has an atom at zero.

The probability density function (pdf) of X is

$$f(t) = \boldsymbol{\alpha}\exp(\mathbf{A}t)\mathbf{a}^0,$$

where $\mathbf{a}^0 = -\mathbf{A}\mathbf{e}'$. The expected value of X can be computed from $E(X) = -\left(\boldsymbol{\alpha}\mathbf{A}^{-1}\mathbf{e}'\right)$. Some well-known continuous phase-type distributions are exponential, Erlang, generalized Erlang, and Coxian distributions.

Phase-type distributions have been widely used in various fields including reliability (Neuts & Meier, 1981) and queueing systems (Yang & Alfa, 2009). Properties and applications of phase-type distributions are well presented in a recent book of He (2014).

The present paper is organized as follows. In Section 2, we define the class C of shock models that we deal with. Section 3 is devoted the optimal replacement time problem. In Section 4, we present a method to compute the mean residual life (MRL) of a system that is defined under a shock model of class C. In Section 5, we define shock models that are members of the class C. Finally in Section 6, we present illustrative examples.

2. The class

Consider a system that is subject to a sequence of shocks over time. Let X_1 denote the time when the first shock occurs; the magnitude of the shock is also assumed as random and is described by a continuous random variable Y_1. Moreover, denote by X_i the interarrival time between the $(i-1)$-th and i-th shocks, and by Y_i the respective magnitude of the i-th shock, $i \geqslant 2$. Assume that both sequences $Y_1, Y_2, \ldots$ and $X_1, X_2, \ldots$ consist of independent and identically distributed random variables. Define a random variable N to be the number of shocks that cause failure of the system. The random variable N can be seen as a stopping random variable and usually is a function of the sequence of random variables $Y_1, Y_2, \ldots$. A shock model M is included in the class C if the system's lifetime under the model M can be written as

$$T = \sum_{i=1}^{N} X_i, \tag{4}$$

where the phase-type random variable N is independent of the sequence of interarrival times $X_1, X_2, \ldots$ that have a common phase-type distribution. In fact, with the class C we consider systems whose lifetimes can be represented as a compound random variable which is a sum of random number of independent random variables. The random variable N counts the number of events, e.g. shocks until the occurrence of a certain event that causes failure of the system. The random variables $X_1, X_2, \ldots$ denote times between events/shocks. The term "shock" should be understood as a potentially harmful event (e.g., high temperature, electrical impulses of large magnitude, etc.). In the extreme shock model, a shock does not have any impact on the system if its magnitude is below a certain level d while it kills the system if the magnitude exceeds or equal to d. In this case, the random variable N represents the number of shocks until the first shock which exceeds or equal to d. Thus the extreme shock model is a member of the class C.

Not all shock models are included in the class C. Consider the δ-shock model. According to the δ-shock model, the system fails when the time between two consecutive shocks is less than a given critical threshold δ (Li & Kong, 2007). In this case,

$$\{N = n\} \text{ iff } \{X_1 > \delta, \ldots, X_{n-1} > \delta, X_n \leqslant \delta\},$$

and $P\{N = n\} = [P\{X_1 > \delta\}]^{n-1} P\{X_1 \leqslant \delta\},\ n = 1, 2, \ldots$ which implies that the random variable N has a phase-type distribution. However, because N depends on the sequence of interarrival times $X_1, X_2, \ldots$, δ-shock model cannot be included in the class C.

As is clear from (4), for a shock model in the class C, the system's lifetime appears as a compound random variable. The rule that describes the failure of the system is defined by the random variable N. As is well-known, the common distribution of the random variables $X_1, X_2, \ldots$ depend on the process in which the shocks occur according to. Obviously, if the shocks occur to a system according to a Poisson process with rate λ, then $P\{X_i \leqslant t\} = 1 - e^{-\lambda t}, t \geqslant 0$.

Assume that $X_1, X_2, \ldots$ are independent and $X_i \sim PH_c(\boldsymbol{\alpha}, \mathbf{A})$, and independently $N \sim PH_d(\mathbf{a}, \mathbf{Q})$. Then

$$T = \sum_{i=1}^{N} X_i \sim PH_c(\boldsymbol{\alpha} \otimes \mathbf{a}, \mathbf{A} \otimes \mathbf{I} + (\mathbf{a}^0\boldsymbol{\alpha}) \otimes \mathbf{Q}), \tag{5}$$

where $\otimes$ is the Kronecker product (He, 2014). That is, phase-type distributions are closed under compounding. Thus the survival function of T can be computed from

$$P\{T > t\} = (\boldsymbol{\alpha} \otimes \mathbf{a})\exp((\mathbf{A} \otimes \mathbf{I} + (\mathbf{a}^0\alpha) \otimes \mathbf{Q})t)\mathbf{e}' \tag{6}$$

which is obvious from (3).

3. Optimal replacement time

A machine or production system may be subject to external shocks in its working environment. Upon its failure, the system or unit is replaced by a new one, and the process repeats. A cost is suffered for each replacement, and additional cost is incurred at each failure in service. Thus, an attempt should be made by an engineer to replace the system before failure. The problem is to find the optimal replacement time which minimizes the total long-run average cost per unit time. In this section, we attack this problem under shock models that are defined by the class C.

According to the classical age replacement policy, the system is replaced upon its failure or upon its reaching age t, whichever occurs first (Ahmad & Kamaruddin, 2012). Let c_1 and c_2 denote respectively the costs of replacing non-failed and failed system. Because a failure incurs an additional penalty, we assume $c_1 < c_2$. If T denotes the lifetime of the system under a shock model, then the mean cost rate per unit time as a function of the replacement age t is given by