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A priori and a posteriori analysis of the meshless Galerkin boundary node method for three-dimensional elasticity

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ABSTRACT

The meshless Galerkin boundary node method is presented in this paper for boundary-only analysis of three-dimensional elasticity problems. In this method, boundary conditions can be implemented directly and easily despite the employed moving least-squares shape functions lack the delta function property, and the resulting system matrices are symmetric and positive definite. A priori error estimates and the consequent rate of convergence are presented. A posteriori error estimates are also provided. Reliable and efficient error estimators and an efficient and convergent adaptive meshless algorithm are then derived. Numerical examples showing the efficiency of the method, confirming the theoretical properties of the error estimates, and illustrating the capability of the adaptive algorithm, are reported.

1. Introduction

Meshless (or meshfree) methods to obtain numerical solutions for partial differential equations without resorting to an element frame have attracted much attention and gained great success in the field of computational science and engineering in the past few decades [1,2]. Compared to the finite element method (FEM) and the boundary element method (BEM), the core of this type of method is to alleviate the difficulty of meshing and remeshing the entire structure by simply adding or deleting nodes. Although many kinds of meshless methods have been proposed, these methods can be simply divided into the domain-type and the boundary-type. Several domain-type meshless methods, such as the element free Galerkin (EFG) method [1–3], the reproducing kernel particle method (RKPM) [1], the point interpolation method (PIM) [2], the generalized FEM [4,5], the h–p meshless method [1], the smoothed FEM [6] and the finite point method (FPM) [7] have been proposed and gained noticeable progress in solving a wide range of boundary value problems.

Boundary integral equations (BIEs) are attractive computational techniques for linear and exterior problems as they can reduce the dimensionality of the original problem by one. Boundary-type meshless methods are developed by the combination of the meshless idea with BIEs, such as the boundary node method (BNM) [8,9], the hybrid BNM [10–13] and the boundary face method [14]. In the three methods, the moving least-squares (MLS) approximations are used to generate the shape functions on the boundary of a domain. However, because MLS shape functions lack the delta function property, boundary conditions in these meshless methods cannot be implemented as easily as in the FEM or the BEM. The strategy employed in them to impose boundary conditions involves a new definition of the discrete norm used for the construction of the MLS approximations, which adds to the number of system equations. In order to construct meshless shape functions with delta function properties, Liu [2] has introduced the PIM into BIEs and produced boundary PIMs. Besides, Li et al. [15] have introduced the radial basis point interpolation into the hybrid displacement variational formulation and produced the hybrid radial BNM. Moreover, Peng and Cheng [16] have developed an improved MLS approximation that uses weighted orthogonal polynomials as basis functions to restore the delta function property of the MLS. The improved MLS scheme has been inserted into BIEs to propose a boundary-type meshless method called the boundary element-free method [16].

Li and Zhu [17] have recently proposed a boundary-type meshless method called the Galerkin boundary node method (GBNM). The GBNM combines a variational version of BIEs for governing equations with the MLS approximations for generation of the trial and test functions. Unlike other MLS-based methods mentioned above, boundary conditions in the GBNM do not present any difficulty and can be implemented with ease via multiplying the MLS shape function and integrating on the boundary. Another outstanding feature of the GBNM is the conservation of the symmetry and positive definiteness of

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the variational problems in the process of numerical implementation. This method has been successfully tried for 2D problems in potential theory [17,18], elasticity [19] and fluid mechanics [20,21]. Very recently, the method has been extended to solve 3D problems in potential theory [22] and fluid mechanics [23]. The present paper extends the frontiers of the GBNM into solving problems in 3D elasticity.

A priori error estimates, which ensure convergence of numerical methods, are crucial in meshless research. The associated mathematical proofs guarantee that meshless methods will converge to the true solution. During the past two decades, a large amount of research has been devoted to deriving a priori error estimation for domain-type meshless methods such as the h–p meshless method [1], the RKPM [1], the EFG [3], the generalized FEM [4], the smoothed FEM [2,6] and the FPM [7]. Nevertheless, although boundary-type meshless methods perform very well in practice, not much is rigorously known on the a priori error analysis of these schemes. Until now, a first a priori error analysis of the boundary-type meshless methods was given by Li and Zhu for the GBNM for 2D problems [17–21] and for 3D problems in potential theory [22] and fluid mechanics [23]. Hence, one goal of this paper is to establish a rigorous a priori error analysis of the GBNM for 3D elasticity problems. The optimal asymptotic convergence rates are given in Sobolev spaces.

In meshless methods, since no predefined nodal connectivity or mesh is employed for field variable approximation, nodes can be inserted or removed conveniently. This prominent feature makes meshless methods especially suited for self-adaptive techniques. Indeed, the subject of a posteriori error estimates and corresponding adaptive procedures is central to the effective application of meshless methods for practical engineering computation. In recent years, a large amount of work has been performed concerning adaptive analysis based on a posteriori error estimation for domain-type meshless methods such as the h–p meshless method [24], the EFG method [2,25], the RKPM [26], the FPM [27] and the PIM [28]. Significant progress has been achieved in the theory and implementation of the adaptive procedures for these meshless methods. For boundary-type meshless methods, Chati et al. [9,29] have pioneered error indicators and an adaptive algorithm for the BNM using hypersingular residual techniques similar to those used in the BEM [9]. Besides, Guo and Chen [30] have developed an adaptive algorithm for the meshless local BIE method based on the dual error indicators.

Very recently, the GBNM has been extended for a posteriori error estimate and adaptivity for 2D potential problems based on the difference between numerical solutions obtained using two successive nodal arrangements [31]. Another aim of this paper is to extend the a posteriori error results to 3D elasticity. The formulation of an accurate and reliable a posteriori error estimate is presented. Then, a posteriori error estimator for the error control of numerical solutions is derived. This error estimator has an upper and a lower bound by the constant multiples of the exact error in the energy norm. That is, this estimator is reliable and efficient. Finally, based on the a posteriori error estimation and a localization technique, computable local error indicators and an efficient and convergent adaptive meshless algorithm for h-adaptivity are established.

The rest of this paper is organized as follows. In Section 2, detailed formulations of the GBNM for 3D elasticity problems are provided. Section 3 contains a priori error analysis and numerical examples showing the performance of the GBNM. In Section 4, a posteriori error analysis and adaptive algorithm are given. Numerical examples illustrating the capability of the adaptive algorithm are also presented in this section. Finally, conclusions are summarized in Section 5.

2. The GBNM for 3D elasticity

Consider the following 3D elasticity problem:

$$\begin{cases} \nabla\boldsymbol{\sigma}=0 & \text{in } \Omega \\ \mathbf{u}|_{\Gamma}=\overline{\mathbf{u}} & \text{on } \Gamma \end{cases} \tag{1}$$

where ∇ is the gradient operator, $\boldsymbol{\sigma}$ is the stress tensor, Ω is a bounded or unbounded domain in $\mathbb{R}^3$ with boundary surface Γ, $\mathbf{u}=(u_1,u_2,u_3)$ is the displacement field, and $\overline{\mathbf{u}}=(\overline{u}_1,\overline{u}_2,\overline{u}_3)$ is a given function on Γ. A general point of Ω is denoted by $\mathbf{x}=(x_1,x_2,x_3)$. If Γ is a smooth open surface piece with a piecewise smooth boundary curve and $\Omega=\mathbb{R}^3\backslash\overline{\Gamma}$, then problem (1) is the so-called screen crack problem.

Let $\overline{\mathbf{u}}$ be given satisfying $\overline{\mathbf{u}}\in\mathbf{H}^{-1/2}(\Gamma):=(H^{-1/2}(\Gamma))^3$, then problem (1) has a unique solution $\mathbf{u}$. The solution can be expressed as [32,33]

$$u_j(\mathbf{y})=\sum_{i=1}^{3}\int_{\Gamma}q_i(\mathbf{x})U_{ij}(\mathbf{x},\mathbf{y})\,\mathrm{d}S_{\mathbf{x}},\quad j=1,2,3,\quad \mathbf{y}\in\Omega \tag{2}$$

in which $\mathbf{q}=(q_1,q_2,q_3)$ is the jump of the boundary traction $\mathbf{n}\cdot\boldsymbol{\sigma}$ across Γ, $\mathbf{n}=(n_1,n_2,n_3)$ is the normal exterior to Γ, and U_{ij} is the singular Kelvin fundamental solution

$$U_{ij}(\mathbf{x},\mathbf{y}):=\frac{1}{16\pi G(1-\nu)r}[(3-4\nu)\delta_{ij}+r_{,i}r_{,j}],\quad i,j=1,2,3 \tag{3}$$

where G is the shear modulus, ν is the Poisson ratio, δ_{ij} is the Kronecker symbol, $r=|\mathbf{x}-\mathbf{y}|$ and $r_{,i}=(x_i-y_i)/r$. Eq. (2) gives the indirect integral equations of 3D elasticity. By direct method, we can also get the direct integral equations and dual integral [34,35].

Using Eq. (2) and the boundary condition of problem (1) leads to the following BIEs:

$$(\mathcal{A}\mathbf{q})_j(\mathbf{y})=\sum_{i=1}^{3}\int_{\Gamma}q_i(\mathbf{x})U_{ij}(\mathbf{x},\mathbf{y})\,\mathrm{d}S_{\mathbf{x}}=\overline{u}_j(\mathbf{y}),\quad j=1,2,3,\quad \mathbf{y}\in\Gamma \tag{4}$$

which are suitable for the solution of the exterior as well as the interior problem. Here the boundary integral operator $\mathcal{A}:\mathbf{H}^{-1/2}(\Gamma)\to\mathbf{H}^{1/2}(\Gamma)$ is continuous and bijective. These BIEs have a unique solution in $\mathbf{H}^{-1/2}(\Gamma)$ and admit the variational problem: Find $\mathbf{q}\in\mathbf{H}^{-1/2}(\Gamma)$ such that

$$\langle\mathcal{A}\mathbf{q},\mathbf{q}'\rangle_{L^2(\Gamma)}=\langle\overline{\mathbf{u}},\mathbf{q}'\rangle_{L^2(\Gamma)},\quad \forall\mathbf{q}'=(q_1',q_2',q_3')\in\mathbf{H}^{-1/2}(\Gamma) \tag{5}$$

where

$$\langle\mathcal{A}\mathbf{q},\mathbf{q}'\rangle_{L^2(\Gamma)}:=\sum_{i,j=1}^{3}\int_{\Gamma}\int_{\Gamma}q_i(\mathbf{x})U_{ij}(\mathbf{x},\mathbf{y})q_j'(\mathbf{y})\,\mathrm{d}S_{\mathbf{x}}\,\mathrm{d}S_{\mathbf{y}}$$

$$\langle\overline{\mathbf{u}},\mathbf{q}'\rangle_{L^2(\Gamma)}:=\sum_{j=1}^{3}\int_{\Gamma}\overline{u}_j(\mathbf{y})q_j'(\mathbf{y})\,\mathrm{d}S_{\mathbf{y}} \tag{6}$$

To carry out integrations in the variational problem (5), the boundary surface Γ is discretized into cells. Assume that Γ is piecewise smooth and can be partitioned into finitely many smooth pieces Υ_k. Each Υ_k can be considered to be the image of G_k by a smooth bijection φ_k, i.e., $\Upsilon_k=\varphi_k(G_k)$, where G_k is a bounded polygonal domain in $\mathbb{R}^2$. Let $\mathcal{T}_{kh}$ be a triangulation of G_k by triangles. On each triangulation $\mathcal{T}_{kh}$, we construct the β-degree interpolant function of φ_k denoted by φ_{kh}. Then the image of G_k by the mapping φ_{kh} constitutes one piece Υ_{kh} of the surface Γ_h which we take as the background cell. Consequently, Γ_h is a connected parametric surface of degree β. After triangulation, we assume that Γ_h contains N cells Γ_i with an associated triangulation $\mathcal{T}_h:=\{\Gamma_1,\Gamma_2,\ldots,\Gamma_N\}$, where the parameter h denotes the nodal spacing.