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Domain Decomposition Method for Diffuse Optical Tomography problems

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ABSTRACT

The forward problem in DOT can be modeled in a frequency domain as a diffusion equation with Robin boundary conditions. In case of multilayered geometries the forward problem can be treated as a set of coupled equations. In this paper we present the solution for diffuse light propagation in a four-layer spherical model using Boundary Element Method. Additionally, we compare overlapping with non-overlapping Domain Decomposition Methods applied to this problem to improve its efficiency.

1. Introduction

In recent years Optical Tomography (OT) has emerged as a highly active and viable field of research, due to advances in both measurement technology and theoretical and practical understanding of the nature of the image of reconstruction problem. For recent reviews see [1–7]. An increasingly active topic within this field is the development of an efficient and accurate method for calculating the intensity of light internal to, and on the boundary of an object under experimental investigation, sometimes referred to as the *forward problem*.

Existing methods are either deterministic, based on the solutions to governing equations, or stochastic, based on the simulations of the individual scattering and absorption events undertaken by each photon. The former includes analytical expressions based on Green functions [8,9] and numerical methods based on Finite Difference Method (FDM) or Finite Element Methods (FEMs) [10–14]. However, a generally applicable model of the forward problem in three-dimensional space is still not a fully solved problem.

In this paper we introduce another standard technique for the solution of Partial Differential Equations (PDEs) in general geometries: the Boundary Element Method (BEM), which has received substantial attention in numerical modeling of fields [15–21]. The advantages and disadvantages of BEM are well known [22–25] and they will not be repeated here, but instead we will concentrate on some specific features useful in OT. Recently BEM has been used in Diffusing-wave spectroscopy for determining the correlation function for different boundary conditions and source properties in a cone-plate geometry [20], but has received very little attention in OT.

Like Finite Element Method (FEM), the Boundary Element Method (BEM) provides a general numerical tool for the solution of complex engineering problems. In the last decades, the range of its applications has remarkably been enlarged.

Nevertheless, the BEM still demands an explicit expression of a fundamental solution, which is only known in simple cases. Therefore, in Optical Tomography BEM is restricted to a diffusion approximation of transport equation.

In recent Optical Tomography applications, researcher's attention is focused on three-dimensional problems. They are a lot more difficult than those defined in the two-dimensional space. Mainly due to the geometry which demands a sophisticated discretization with enormously big number of unknowns. Such problems are named '*large scale problems*'.

BEM is characterized by the boundary-only property of the algorithm. This property reduces the number of unknowns in BEM as compared to those in methods of the domain type such as Finite Difference Method (FDM) or Finite Element Method (FEM).

However, the reduced number of unknowns does not necessarily lead to improved efficiency, because BEM generally produces a fully populated asymmetric matrix of coefficients, while the matrices for FDM or FEM are usually sparse and very often symmetric.

Because of this drawback, BEM has so far been considered to be less efficient than these domain type competitors in large scale problems. However, the situation is changing with the recent breakthrough introduced by the so-called 'fast-BEMs' based on techniques such as multiple methods [26], panel clustering, the use of wavelet bases, etc.

The fast BEMs can compute potential functions at all collocation points with $O(N)–O(N(\log N)^m)$ $(m \geq 0)$ operations in problems with N unknowns.

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This is a dramatic improvement over the conventional BEMs which have $O(N^2)$ number of operations. Development of the fast BEM is certain to further enhance the status of BEM as a solver of large scale problems.

2. Governing equations

Diffuse Optical Tomography in medicine aims to recover the optical properties of biological tissue from the measurement of the transmitted light made at multiple points on the surface of the body. This boundary data measurements can be used to recover the spatial distribution of internal absorbtion and scattering coefficients. It is a non-invasive modality and can generate images of parameters related to blood volume and oxygenation [5].

The main topic within this field is the development of an efficient and accurate method for calculating the intensity of light transmitted or reflected from the object under experimental investigation. A general model of light propagation can be described using the Radiative Transfer Equation, but a simpler model that can be derived from this equation in the case of sufficiently high scattering is the diffusion equation with Robin boundary conditions [27].

In this paper it is assumed that the object being studied is considered as a set of disjoint simply connected regions with constant optical coefficients within each region, but that may differ between regions. In this case the diffusion equation can be replaced by a set of Helmholtz equations for each domain, together with interface conditions. For this problem, analytical solution is not easily available. Although volume based PDE solvers such as FDM or FEM can certainly be applied to this problem, there are often practical difficulties in constructing meshes for general geometries that respect the interfaces accurately. In contrast, the use of boundary integral methods (e.g. BEM) involves only representation of the surface meshes and can be much easier to implement.

The problem of Optical Tomography in a highly diffusive body Ω with boundary Γ can be modeled by the use of the diffusion equation in the frequency domain form [4,9]

$$-\nabla\cdot\kappa(\mathbf{r})\nabla\Phi(\mathbf{r};\omega)+\mu_a(\mathbf{r})\Phi(\mathbf{r};\omega)+\frac{(i\omega)}{c}\Phi(\mathbf{r},\omega)=q(\mathbf{r};\omega), \tag{1}$$

with Robin boundary conditions

$$\Phi(\mathbf{m};\omega)+2\alpha\kappa(\mathbf{m})\frac{\partial\Phi(\mathbf{m};\omega)}{\partial n}=h^-(\mathbf{m};\omega),\quad \mathbf{m}\in\Gamma, \tag{2}$$

where $\omega\in\mathbf{R}^+$ is the frequency modulation, Φ is the photon density, c is the velocity of light, q is an internal source of light in medium, h^- is an incoming flux, α is a boundary term which incorporates the refractive index mismatch at the tissue–air boundary, $\mathbf{n}$ is the outward normal at the boundary Γ, κ and μ_a are the diffusion and absorption coefficients, respectively. We define, $\kappa=1/3(\mu_a+\mu_s')$, where μ_s' is the reduced scattering coefficient [28,9]. We use the notation $\mathbf{r}$ for a position vector in Ω and $\mathbf{m}$ for a position vector restricted to the surface Γ.

3. Singular and nearly singular integrals

In three-dimensional boundary element analysis, computation of integrals is an important aspect since it governs the accuracy of the analysis and also because it usually takes the substantiable part of the CPU time.

The integrals which determine the influence matrices, the internal field and its gradients contain nearly singular kernels of order $1/R^\alpha$ $(\alpha=1,2,3,4,\ldots)$ where R is the distance between the source point and the integration point on the boundary element [29].

For planar elements, analytical integration may be possible [17]. However, it is becoming increasingly important, in practical boundary element codes, to use curved elements, such as the isoparametric elements, to model general curved surfaces [30]. Since analytical integration is not possible for general isoparametric curved elements, one has to rely on numerical integration.

When the distance between the source point and the element over which the integration is performed is sufficiently large, compared to the element size, the standard Gauss–Legendre quadrature formula works efficiently.

However, when the source is actually on the element, the kernel becomes singular and the straight forward application of the Gauss–Legendre quadrature formula breaks down. These integrals will be called singular integrals. Singular integrals occur when calculating the diagonals of the coefficient matrix.

When the source is not on the element, but very close to the element, although the kernel is regular in the mathematical sense, the value of the kernel changes rapidly in the neighborhood of the source point. In such case the standard Gauss–Legendre quadrature formula is not practical, since it would require a huge number of integration points to achieve the required accuracy.

These integrals will be called *nearly singular integrals*. Nearly singular integrals occur in practice when calculating influence matrices for thin structures, where distances between different elements can be very small compared to the element size. Such situation is very common for scull or CSF layer modeling in Impedance or Optical Tomography. They also occur when calculating the field or its derivatives at the internal point, very close to the boundary element. Numerous research works have already been published on this subject for example [31–34,19,29,16].

In this paper the coordinate transformation methods are used (see for example Section 6). This method is transforming a triangular region (if the triangular boundary elements are used) into a quadrilateral region, so that the node corresponding to the singularity is expanded to an edge of the quadrilateral, thanks to that the singularity is weakened.

Nearly singular integrals turn out to be more difficult and expensive to calculate compared to singular integrals. They are becoming more and more important in practical boundary element codes, since the ability and efficiency to calculate nearly singular integrals governs the code's versatility in treating objects containing thin structures (scull or CSF layer of the human head). The reader interested in this particular problem may consult [29] where a new quadrature scheme for the accurate and efficient evaluation of these nearly singular integrals is presented.

4. Curvilinear triangular boundary element

To study boundary elements which are two-dimensional structures placed in the 3D space, first we need to define the way in which we can pass from the xyz global Cartesian system to the ξ_1,ξ_2,ξ_3 system defined over the element, where ξ_1,ξ_2 are oblique coordinates and ξ_3 is in the direction of the normal. The transformation for a given function Φ is related through the following:

$$\begin{bmatrix}\frac{\partial\Phi}{\partial\xi_1}\\ \frac{\partial\Phi}{\partial\xi_2}\\ \frac{\partial\Phi}{\partial\xi_3}\end{bmatrix}=\begin{bmatrix}\frac{\partial x}{\partial\xi_1} & \frac{\partial y}{\partial\xi_1} & \frac{\partial z}{\partial\xi_1}\\ \frac{\partial x}{\partial\xi_2} & \frac{\partial y}{\partial\xi_2} & \frac{\partial z}{\partial\xi_2}\\ \frac{\partial x}{\partial\xi_3} & \frac{\partial y}{\partial\xi_3} & \frac{\partial z}{\partial\xi_3}\end{bmatrix}\begin{bmatrix}\frac{\partial\Phi}{\partial x}\\ \frac{\partial\Phi}{\partial y}\\ \frac{\partial\Phi}{\partial z}\end{bmatrix}, \tag{3}$$

where the square matrix is the Jacoby matrix.

Transformation of this type allows us to describe differentials of surface in the Cartesian system in terms of the curvilinear coordinates. A differential of area will be given by

$$d\Gamma=|\mathbf{n}|\,d\xi_1\,d\xi_2=\left|\frac{\partial\mathbf{r}}{\partial\xi_1}\times\frac{\partial\mathbf{r}}{\partial\xi_2}\right|d\xi_1\,d\xi_2 = \sqrt{n_x^2+n_y^2+n_z^2}\,d\xi_1\,d\xi_2, \tag{4}$$