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A non linear approximation method for solving high dimensional partial differential equations: Application in finance

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Abstract

We study an algorithm which has been proposed by A. Ammar, B. Mokdad, F. Chinesta, R. Keunings in 2006 to solve high-dimensional partial differential equations. The idea is to represent the solution as a sum of tensor products and to compute iteratively the terms of this sum. This algorithm is related to the so-called greedy algorithms, as introduced by V.N. Temlyakov. In this paper, we investigate the application of the greedy algorithm in finance and more precisely to the option pricing problem. We approximate the solution to the Black–Scholes equation and we propose a variance reduction method. In numerical experiments, we obtain results for up to 10 underlyings. Besides, the proposed variance reduction method permits an important reduction of the variance in comparison with a classical Monte Carlo method.

Keywords: Greedy algorithms; Black–Scholes partial differential equation; Variance reduction

Introduction

Many problems of interest for various applications (for example material sciences and finance) involve high-dimensional partial differential equations (PDEs). The typical example in finance is the pricing of a basket option, which can be obtained by solving the Black–Scholes PDE with dimension the number of underlying assets.

We propose to investigate an algorithm which has been proposed by Chinesta et al. [2] for solving high-dimensional Fokker–Planck equations in the context of kinetic models for polymers, and by Nouy et al. [8] in uncertainty quantification framework based on previous works by Ladevèze [6]. This approach is also studied in [7] to try to circumvent the curse of dimensionality for the Poisson problem in high-dimension. This approach is a nonlinear approximation method called below the *greedy algorithm* because it is related to the so-called greedy algorithms introduced in nonlinear approximation theory, see for example [10]. The main idea is to represent the solution as a sum of tensor products (referred to as a *separated representation* in the following):

$$\begin{aligned} u(x_1,\ldots,x_d) &= \sum_{k\geq 1} r_k^1(x_1) r_k^2(x_2)\ldots r_k^d(x_d) \\ &= \sum_{k\geq 1} \left(r_k^1 \otimes r_k^2 \otimes \cdots \otimes r_k^d \right)(x_1,\ldots,x_d) \end{aligned} \tag{1}$$

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and to compute iteratively each term of this sum using a greedy algorithm. This greedy algorithm can be applied to any PDE which admits a variational interpretation as a minimization problem. The practical interest of this algorithm has been demonstrated in various contexts (see for example [2] for applications in fluid mechanics).

Our contribution is to complete the first application of this algorithm in finance, investigating the interest of this approach for option pricing. In this work, our aim is to study the problem of pricing vanilla basket options of European type using two numerical methods: first a discretization technique for the Black–Scholes PDE and a variance reduction method for the pricing of the same type of financial products.

For option pricing, we will discuss in particular the key points to be solved to address problems in finance, compared to the situations studied in [7] or in [2], that is, the treatment of the non zero boundary conditions and the approximation of the solution to the Black–Scholes PDE as a sequence of minimization problems. We will study also the practical implementation of the algorithm. We will not solve the minimization problems associated to the PDE, but the first-order optimality conditions of these minimization problems, namely the Euler equation. This leads to a system of equations where the number of degrees of freedom does not grow exponentially with respect to the dimension, and this fact will be very important in order to attain high-dimensional frameworks in practical applications. More precisely, the Euler equation writes as a system of d nonlinear equations, where d is the considered dimension. The maximum dimension that can be treated by this technique is limited by the *non-linearity* of the system of d equations that has to be solved.

The variance reduction method relies on the backward Kolmogorov equation which yields an exact control variate. We propose to solve the high-dimensional Kolmogorov equation using the greedy algorithm. This yields an efficient pricing method which combines deterministic and stochastic techniques.

We would like to point out that in order to circumvent the curse of dimensionality using the greedy algorithm, the initial condition of the Black–Scholes equation has to be set out in separated representation with respect to the different coordinates, namely a sum of tensor products. As the initial condition is not always expressed in a separated representation, we first need to investigate the problem of approximating the initial condition by a sum of tensor products. This problem will be solved using again the greedy algorithm. We will provide examples to illustrate that this approximation is suitable.

Other deterministic techniques have been applied to solve the Black–Scholes PDE in a high-dimensional framework. Classical methods such as finite differences and finite elements are limited in their application when the dimension increases (typically $d \leq 4$), because the number of degrees of freedom increases exponentially with respect to the dimension and rapidly exceed the limited storage capacity. Financial applications of the sparse tensor product methods have been studied by Pommier in [9]. These sparse methods also use the representation of the solution as a sum of tensor products, and assume that the solution is regular enough to obviate fine discretizations in each direction.

In our numerical experiments, the greedy algorithm gives results for up to 10 underlyings; that means the dimension d is equal to 10. To the best of our knowledge, this is higher than results obtained using other deterministic approaches, such as the tensor product method, for which examples up to the dimension $d = 5$ have been reported in the literature, see [9]. In addition, the variance reduction method that we are proposing permits the variance to be reduced in comparison with a classic Monte Carlo method.

The plan of the paper is the following. In Section 1, we introduce the general setting for the greedy algorithm and we give some theoretical results that have been proved in the literature and that ensure the convergence of the greedy algorithm. Section 2.1 will discuss the practical implementation of the greedy algorithm in the case of the approximation of a function by a sum of tensor products. Following this, in Sections 2.2 and 2.3, we will present results and applications for the approximation of a basket put option. The purpose of Section 3 is to apply the greedy algorithm to solve the Black–Scholes PDE. After introducing the weak formulation of this equation in Section 3.1, we treat the difficulties that arise when applying the greedy algorithm such as posing the problem in a bounded domain (Section 3.2) and recasting the PDE as a minimization problem (Section 3.3). The final section (Section 4) contains numerical results for the solution of the Black–Scholes PDE and for the variance reduction method.

1. Greedy algorithms for high dimensional problems

In this section, we define a general framework for the greedy algorithm that we will use to solve the high-dimensional problems studied in this paper.