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Integration and approximation in cosine spaces of smooth functions

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Abstract

We study multivariate integration and approximation for functions belonging to a weighted reproducing kernel Hilbert space based on half-period cosine functions in the worst-case setting. The weights in the norm of the function space depend on two sequences of real numbers and decay exponentially. As a consequence the functions are infinitely often differentiable, and therefore it is natural to expect exponential convergence of the worst-case error. We give conditions on the weight sequences under which we have exponential convergence for the integration as well as the approximation problem. Furthermore, we investigate the dependence of the errors on the dimension by considering various notions of tractability. We prove sufficient and necessary conditions to achieve these tractability notions.

Keywords: Numerical integration; Function approximation; Cosine space; Worst-case error; Exponential convergence-Tractability

1. Introduction

In this paper we study two instances of multivariate linear problems. We are interested in approximating linear operators $S_s : \mathcal{H}_s \to \mathcal{G}_s$, where $\mathcal{H}_s$ is a certain Hilbert space of s-variate functions defined on $[0,1]^s$ and where $\mathcal{G}_s$ is a normed space, namely:

- Numerical integration of functions $f \in \mathcal{H}_s$: In this case, we have $\mathcal{G}_s = \mathbb{R}$ and $S_s(f) = \mathrm{INT}_s(f) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$;
- L_2-approximation of functions $f \in \mathcal{H}_s$: In this case, we have $\mathcal{G}_s = L_2([0,1]^s)$ and $S_s(f) = \mathrm{APP}_s(f) = f$.

Without loss of generality, see, e.g., [14] or [11, Section 4], we approximate S_s by linear algorithms $A_{n,s}$ using n function evaluations of the form

$$A_{n,s}(f) = \sum_{j=1}^{n} \alpha_j f(\boldsymbol{x}_j) \quad \text{for all } f \in \mathcal{H}_s, \tag{1}$$

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where $\alpha_j \in \mathcal{G}_s$ and $\boldsymbol{x}_j \in [0,1]^s$ for all $j = 1, 2, \ldots, n$. That is, we consider only algorithms using the class of standard information. For multivariate approximation, it also makes sense to consider the information class Λ^{all} which contains not only function evaluations but all linear functionals, see, e.g. [8].

We measure the error of an algorithm $A_{n,s}$ in terms of the *worst-case error*, which is defined as

$$e(A_{n,s}, S_s) := \sup_{\substack{f \in \mathcal{H}_s \\ \|f\|_{\mathcal{H}_s} \le 1}} \left\| S_s(f) - A_{n,s}(f) \right\|_{\mathcal{G}_s},$$

where $\|\cdot\|_{\mathcal{H}_s}$, $\|\cdot\|_{\mathcal{G}_s}$ denote the norms in $\mathcal{H}_s$ and $\mathcal{G}_s$, respectively. The *nth minimal (worst-case) error* is given by

$$e(n, S_s) := \inf_{A_{n,s}} e(A_{n,s}, S_s),$$

where the infimum is taken over all admissible algorithms $A_{n,s}$. For $n = 0$, we consider algorithms that do not use information evaluations and therefore we use $A_{0,s} \equiv 0$. The error of $A_{0,s}$ is called the *initial (worst-case) error* and is given by

$$e(0, S_s) := \sup_{\substack{f \in \mathcal{H}_s \\ \|f\|_{\mathcal{H}_s} \le 1}} \|S_s(f)\|_{\mathcal{G}_s} = \|S_s\|.$$

During the past years many examples of weighted reproducing kernel Hilbert spaces of functions with exponentially decaying weights have been studied, for example Korobov spaces of one-periodic functions on the unit cube whose Fourier coefficients decay exponentially fast (see [1,2,9,10]) or Hermite spaces of functions on $\mathbb{R}^s$ whose Hermite coefficients decay exponentially fast (see [6,8,7]). The recent paper [8] studies multivariate approximation over Hilbert spaces with exponential weights by linear algorithms based on arbitrary linear functionals (in other words, the algorithms are allowed to use the information class Λ^{all}) in a very general setting which covers Korobov and Hermite spaces in the aforementioned sense, but also so-called Walsh spaces and cosine spaces. The advantage of the cosine space over the Korobov space is that in this setting one gets rid of the periodicity assumption.

In all of these papers it is shown that one is able to obtain errors that converge to zero very quickly as n increases, namely exponentially fast. By *exponential convergence* (EXP) of the worst-case error we mean that there exist a number $q \in (0,1)$ and functions $p, C, M : \mathbb{N} \to (0, \infty)$ such that

$$e(n, S_s) \le C(s)\, q^{\,(n/M(s))^{\,p(s)}} \quad \text{for all } s, n \in \mathbb{N}. \tag{2}$$

If the function p in (2) can be taken as a constant function, i.e., $p(s) = p > 0$ for all $s \in \mathbb{N}$, we say that we achieve *uniform exponential convergence* (UEXP) of $e(n, S_s)$. Furthermore, we denote by $p^*(s)$ and p^* the largest possible rates $p(s)$ and p such that EXP and UEXP hold, respectively.

When studying algorithms $A_{n,s}$, we do not only want to control how their errors depend on n, but also how they depend on the dimension s. This is of particular importance for high-dimensional problems. To this end, we define, for $\varepsilon \in (0,1)$ and $s \in \mathbb{N}$, the *information complexity* by

$$n(\varepsilon, S_s) := \min\{n : e(n, S_s) \le \varepsilon\, e(0, S_s)\}$$

as the minimal number of information evaluations needed to reduce the initial error by a factor of ε. EXP implies that asymptotically we need $\mathcal{O}(\log^{1/p(s)} \varepsilon^{-1})$ information evaluations for $\varepsilon \to 0$, to compute an ε-approximation. However, it is not clear how long we have to wait to see this nice asymptotic behavior especially for large s. This is the subject of tractability. Thus, we intend to study how the information complexity depends on $\log \varepsilon^{-1}$ and s by considering the following tractability notions, which were already considered in [1,2,6,8–10]. We say that we have *Exponential Convergence-Weak Tractability (EC-WT)* if

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log\, n(\varepsilon, S_s)}{s + \log\, \varepsilon^{-1}} = 0$$

with the convention $\log\, 0 = 0$, i.e., we rule out the cases for which $n(\varepsilon, s)$ depends exponentially on s and $\log\, \varepsilon^{-1}$. If there exist numbers $c, \tau, \sigma > 0$ such that

$$n(\varepsilon, S_s) \le c\, s^{\,\sigma}\, (1 + \log\, \varepsilon^{-1})^{\,\tau} \quad \text{for all } s \in \mathbb{N},\ \varepsilon \in (0,1), \tag{3}$$