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Study of new rare event simulation schemes and their application to extreme scenario generation

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Abstract

This is a companion paper based on our previous work on rare event simulation methods. In this paper, we provide an alternative proof for the ergodicity of shaking transformation in the Gaussian case and propose two variants of the existing methods with comparisons of numerical performance. In numerical tests, we also illustrate the idea of extreme scenario generation based on the convergence of marginal distributions of the underlying Markov chains and show the impact of the discretization of continuous time models on rare event probability estimation.

Keywords: Extreme scenario; Rare event; Markov chains; Ergodic properties; Interacting particle systems

1. Introduction

Our aim is to estimate quantities such as $\mathbb{P}(X \in A)$, $\mathbb{E}(\varphi(X)|X \in A)$ and $\mathbb{E}(\varphi(X)1_{X\in A})$ where X is a random variable which takes value in a measurable state space $\mathcal{X}$ and $A \subset \mathcal{X}$ is a small subset. Another problem of interest is to sample from the conditional distribution $X|X \in A$, i.e. the generation of extreme scenarios. These quantities readily appear in the context of reliability for various industrial fields such as communication systems, aircraft safety, nuclear reactors safety, and so forth. Furthermore, they are also related to financial risks, for example model, credit and actuarial risk. Notably, the problem of extreme scenario generation is known as stress-testing in the field of financial risk management. The current methodology is to introduce artificial market specific and idiosyncratic shocks in the model and then to compute the default probabilities/the expected loss. These handpicked scenarios are sometimes simplified and can misrepresent risks in different ways [3] and, in a dynamic setting, can fail to capture the interdependence between various stochastic factors in the market (see Section 4.2 in [6]). In order to study the

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stability of a financial – say, banking – system, it is essential to generate extreme scenarios given the bank/the system default. In this work (Section 3), we present some examples where this is achieved.

In the case of a rare event, where the probability $\mathbb{P}(X \in A)$ is extremely small, simple Monte Carlo method is inefficient. Alternatively, we can use Importance Sampling methods which can be very efficient but require information about the model and the rare event. The schemes based on Interacting Particle System (IPS) or splitting techniques or ergodic transformations are also well developed. These approaches can be combined. For a detailed description, see [7]. We elaborate here on the splitting and shaking transformation approach designed therein. To implement this approach, we first create a nested sequence of more and more rare events:

$$\mathcal{X} = A_0 \supset \cdots \supset A_k \supset \cdots \supset A_n = A.$$

Then, we can write the quantities above via the following products:

$$\mathbb{P}(X \in A) = \prod_{k=1}^{n} \mathbb{P}(X \in A_k | X \in A_{k-1}), \tag{1.1}$$

$$\mathbb{E}(\varphi(X)\mathbf{1}_{X\in A}) = \mathbb{E}(\varphi(X)|X \in A_n) \prod_{k=1}^{n} \mathbb{P}(X \in A_k | X \in A_{k-1}). \tag{1.2}$$

Thus, instead of estimating the rare event probability directly, we estimate each conditional probability (and expectation) on the right hand side. Two different schemes, Parallel-One-Path (POP) and Interacting Particle System (IPS), using reversible shaking transformation, are proposed in [7] and further studied in [1]. POP is based on the time average of a Markov chain while IPS is based on the space average of a large Markovian particle system. As is shown in the above references, the shaking transformation is the tool which unifies IPS and POP methods. A similar approach to POP has been developed in the engineering sciences community (see [2]), and is called in that context *subset simulation* approach.[1] We emphasize that the shaking transformation designed in [7] avoids using explicit transition kernels and is presented with a simple parametrization. It facilitates quick implementation and makes it easier to tune shaking forces. Our framework can also handle general situations including stochastic processes driven by Brownian motion or Poisson process. Furthermore, we have developed an adaptive version of the POP method and proved its consistency in [1].

In this paper, our aim is threefold:

- We provide efficient variants of POP and IPS methods. The IPS algorithm is revisited by incorporating extra resampling to enhance the independence property between particles and by reducing the size of the particle system to keep the same computational cost. The adaptive POP algorithm is modified to preserve the benefits of both adaptivity and parallelization.
- When dealing with stochastic processes, we test the robustness of the two algorithms w.r.t. the dimension of the discretization of the underlying noise (i.e. the number of Brownian increments) in order to address the issue of the curse of dimensionality. As a difference with usual MCMC algorithms where the dimension effect may be significant (see for example [8]), we do not observe any significant decrease of speed of convergence.
- We illustrate the application of these methods for extreme (rare-event) scenario generation.

2. Parallel-one-path (POP) and interacting particle system (IPS) method

2.1. Shaking transformation

POP and IPS methods are designed to estimate each term on the r.h.s. of (1.1) using a random transformation called *shaking transformation* (a local distribution-preserving perturbation). For a given random variable $X \in \mathcal{X}$, $\mathcal{K}(\cdot)$ is called a shaking transformation for X if $(X, \mathcal{K}(X)) \stackrel{d}{=} (\mathcal{K}(X), X)$. We will consider the case where $\mathcal{K}(X) = K(X, Y)$, where $K(\cdot, \cdot)$ is a deterministic function and Y is another random variable independent of X. Then, we can build a $\mathcal{X}$-valued Markov chain as

$$X_{i+1} = K(X_i, Y_i), \qquad X_0 \stackrel{d}{=} X,$$

[1] We would like to thank Robert Scheichl for directing our attention to this reference.