Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 143 (2018) 149–157
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

A discrete stochastic Gronwall lemma

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Received 27 January 2016; received in revised form 30 June 2016; accepted 5 July 2016
Available online 14 July 2016

Abstract

The purpose of this paper is the derivation of a discrete version of the stochastic Gronwall lemma involving a martingale. The proof is based on a corresponding deterministic version of the discrete Gronwall lemma and an inequality bounding the supremum in terms of the infimum for discrete time martingales. As an application the proof of an a priori estimate for the backward Euler–Maruyama method is included.

Keywords: Gronwall lemma; Martingale inequality; Backward Euler–Maruyama method; A priori estimate

1. Introduction

The Gronwall lemma is an often used tool in classical analysis for deriving a priori and stability estimates of solutions to differential equations. It is named after T.H. Grönwall and originated in its differential form from his work [6]. Besides the integral version in [4] many more variations of the Gronwall lemma have been introduced. Not surprisingly, we find applications of Gronwall lemmas frequently, for example, in the study of ordinary differential equations, partial differential equations, integral equations, and stochastic analysis. Discrete versions are often applied in order to estimate the growth of solutions to difference equations, for example, those originating from numerical approximations of differential equations. For instance, we refer to [3,5] and the references therein.

The purpose of this paper is the derivation of the following discrete time and stochastic version of this Gronwall lemma:

Theorem 1. *Let $(M_n)_{n\in\mathbb{N}_0}$ be an $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$-martingale satisfying $M_0 = 0$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$. Let $(X_n)_{n\in\mathbb{N}_0}$, $(F_n)_{n\in\mathbb{N}_0}$, and $(G_n)_{n\in\mathbb{N}_0}$ be sequences of nonnegative and adapted random variables with $\mathbb{E}[X_0] < \infty$ such that*

$$X_n \le F_n + M_n + \sum_{k=0}^{n-1} G_k X_k, \quad \text{for all } n \in \mathbb{N}_0. \tag{1}$$

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http://dx.doi.org/10.1016/j.matcom.2016.07.002
0378-4754/

Then, for any $p \in (0,1)$ and $\mu, \nu \in [1,\infty]$ with $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and $p\nu < 1$, it holds true that

$$\mathbb{E}\big[\sup_{0\le k\le n} X_k^p\big] \le \Big(1 + \frac{1}{1-\nu p}\Big)^{\frac{1}{\nu}} \Big\| \prod_{k=0}^{n-1} (1+G_k)^p \Big\|_{L^\mu(\Omega)} \Big(\mathbb{E}\big[\sup_{0\le k\le n} F_k\big]\Big)^p \tag{2}$$

for all $n \in \mathbb{N}_0$. In particular, if $(G_n)_{n\in\mathbb{N}_0}$ is a deterministic sequence of nonnegative real numbers, then for any $p \in (0,1)$ it holds true that

$$\mathbb{E}\big[\sup_{0\le k\le n} X_k^p\big] \le \Big(1 + \frac{1}{1-p}\Big) \Big(\prod_{k=0}^{n-1} (1+G_k)^p\Big) \Big(\mathbb{E}\big[\sup_{0\le k\le n} F_k\big]\Big)^p \tag{3}$$

for all $n \in \mathbb{N}_0$.

The main novelty of Theorem 1, whose continuous time counter-part recently appeared in [10], is the presence of a martingale term on the right hand side of Eq. (1).

First, we emphasize that the estimates in Eqs. (2) and (3) are uniform with respect to the martingale $(M_n)_{n\in\mathbb{N}_0}$. The price we have to pay for this uniformity is the restriction of the parameter p to the interval $(0,1)$. If one is interested in estimates for values $p \ge 1$ one could instead try to apply, for instance, Burkholder–Davis–Gundy-type inequalities resulting in the appearance of the quadratic variation of the martingale on the right hand side of the estimates. For a further discussion of the case $p \ge 1$ we also refer to Remark 4.

Second, in order to better illustrate the difference to estimates directly obtained by deterministic versions of the Gronwall lemma, let us for a moment assume the situation of Theorem 1 but with $(G_n)_{n\in\mathbb{N}_0}$ being a deterministic sequence of nonnegative real numbers. Then, if we first take expectation in Eq. (1) we obtain

$$\mathbb{E}[X_n] \le \mathbb{E}[F_n] + \sum_{k=0}^{n-1} G_k \mathbb{E}[X_k]. \tag{4}$$

Here we are in a position to apply a deterministic and discrete time version of the Gronwall lemma, e.g. Lemma 2. For each $n \in \mathbb{N}$ this yields

$$\begin{aligned} \mathbb{E}[X_n] &\le \mathbb{E}[F_n] + \sum_{k=0}^{n-1} \mathbb{E}[F_k] G_k \prod_{j=k+1}^{n-1} (1+G_j) \\ &\le \sup_{0\le k\le n} \mathbb{E}[F_k] \prod_{j=0}^{n-1} (1+G_j). \end{aligned}$$

From this one can deduce the estimate

$$\sup_{0\le k\le n} \mathbb{E}[X_n] \le \sup_{0\le k\le n} \mathbb{E}[F_k] \prod_{j=0}^{n-1} (1+G_j). \tag{5}$$

Comparing the estimates (3) and (5) shows that both estimates are again independent of the martingale $(M_n)_{n\in\mathbb{N}_0}$. The latter estimate is weaker in the sense that taking the supremum with respect to k occurs outside the expectation on the left hand side of Eq. (5). At the same time, Eq. (5) is stronger in the sense that it gives an estimate of $\mathbb{E}[X_n^p]$ in the case of $p = 1$, which is excluded in Eq. (3).

In addition, it is worth noting the following subtle difference between Theorem 1 and its continuous time counter-part in [10]: On the right hand side of Eqs. (2) and (3) we have the pth power of the expectation of $\sup_{0\le k\le n} F_k$. In [10, Theorem 4] the order of the pth power and the expectation is reversed resulting in a sharper estimate. The reason for this difference lies in the martingale inequality in Lemma 3 which for discrete time martingales only holds true in the weaker form used in this paper. Compare further with [10, Remark 3].

The proof of Theorem 1 is mostly based on two ingredients: The first is a discrete version of the classical Gronwall lemma which is found in Lemma 2. The second ingredient is an inequality stated in Lemma 3 that relates the L^p-norm, $p \in (0,1)$, of the supremum of a discrete time martingale to its infimum. Lemma 3 is therefore the discrete time counter-part of [10, Proposition 1]. A further version of the latter with optimal constant is also found in [2]. For all details of the proof we refer to Section 2.