Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 143 (2018) 176–185
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

The non-locality of Markov chain approximations to two-dimensional diffusions

C. Reisinger

Mathematical Institute, University of Oxford, Woodstock Road, Oxford, OX2 6GG, United Kingdom

Received 31 January 2016; received in revised form 20 May 2016; accepted 1 June 2016
Available online 14 June 2016

Abstract

In this short paper, we consider discrete-time Markov chains on lattices as approximations to continuous-time diffusion processes. The approximations can be interpreted as finite difference schemes for the generator of the process. We derive conditions on the diffusion coefficients which permit transition probabilities to match locally first and second moments. We derive a novel formula which expresses how the matching becomes more difficult for larger (absolute) correlations and strongly anisotropic processes, such that instantaneous moves to more distant neighbours on the lattice have to be allowed. Roughly speaking, for non-zero correlations, the distance covered in one timestep is proportional to the ratio of volatilities in the two directions. We discuss the implications to Markov decision processes and the convergence analysis of approximations to Hamilton–Jacobi–Bellman equations in the Barles–Souganidis framework.

Keywords: Two-dimensional diffusions; Markov chain approximations; Monotone finite difference schemes

1. Introduction

The approximation of multi-dimensional continuous-time stochastic processes by Markov chains is a question of practical importance, which leads to a surprisingly complex answer. Our interest is motivated by stochastic control problems, as they arise very prominently in mathematical finance and other application areas.

The computation of optimal control strategies usually relies on the computation of a value function, this being the expected value of an objective under the optimal control strategy, as a function of the starting point. Approximation of this function by dynamic programming is usually either mesh-based or estimated by regression of conditional expectations over simulated trajectories. We focus here on the former approach.

A classical strategy consists in the approximation of the continuous-time stochastic process by a discrete-time Markov chain on a lattice, with a certain time step and mesh width of the lattice, as in the standard Ref. [15]. Then it can be shown, under relatively mild assumptions on the underlying diffusions, that if the moments are matched locally in time and state space, asymptotically for vanishing time step and mesh width, the value of the control problem associated with the discrete process converges to the original value function.

E-mail address: christoph.reisinger@maths.ox.ac.uk.

http://dx.doi.org/10.1016/j.matcom.2016.06.001

These approximations are intimately linked to finite difference schemes for the Hamilton–Jacobi–Bellman PDEs which govern the value function. Indeed, a standard method of deriving suitable transition probabilities for the Markov chains is based on finite difference approximations to the generator of the process. The moment matching conditions translate into consistency of the corresponding finite difference schemes. It is evident that only those finite difference schemes lead to viable Markov chain approximations, where the elements of the discretisation matrix, a scaled version of which are the candidate transition probabilities, are non-negative.

Barles and Souganidis [4] prove that under certain mild regularity conditions and a comparison principle, a consistent, monotone and stable approximation scheme converges to the true viscosity solution of a second-order non-linear PDE. Since then, many papers have added to the analysis, including convergence orders, of monotone approximation schemes especially for equations of Hamilton–Jacobi–Bellman(HJB)-type, see, e.g., [13,6,1–3,9]. The requirement of monotonicity is equivalent to a condition on positivity of the coefficients (see [11]), thus demonstrating a one-to-one correspondence between those finite difference scheme with generally provable convergence, and Markov chain approximations.

The focus of this paper is the construction of transition probabilities, or – equivalently – monotone discretisation operators, in the two-dimensional case. We now review prior results from the literature, often presented in the context of a (degenerate) elliptic PDE and we translate this into our language.

It is already recognised in [15] that the approximation becomes more difficult when the off-diagonal terms of the covariance matrix are comparable to or larger than the diagonal terms. This is the case for large correlations and anisotropic variances.

More specifically, a result tracing back to [18] implies that a positive coefficient discretisation using only neighbouring nodes exists if and only if the covariance matrix is diagonally dominant. Moreover, [8] show that if the problem is degenerate (correlation $\rho = 1$), then the problem can be approximated locally, i.e., using transitions to mesh points which are a finite number of steps away, if and only if the ratio between the volatilities in both directions is rational. This will also be directly visible from our analysis. Such equations with perfect correlation arise, e.g., for stochastic control problems with partial information [10, Chapter 6, Section 10]. Specifically, [17] studies financial derivative hedging and valuation in incomplete markets with unknown (constant) drift of the asset price processes. After application of the Kálmán–Bucy filter, the process for the filtered drift is driven by the same Brownian motion as the asset price process, resulting in degenerate multi-dimensional diffusions. The present results can therefore be of interest in this setting.

In the case where monotone schemes exist in the irrational but non-degenerate case, [12] derives lower and upper bounds on the required minimal stencil size using Diophantine equations and approximation results for irrationals by rationals. Upper and lower bounds are given in terms of inverse fractional powers of the parameter $1 - |\rho|$, demonstrating that the approximation schemes become necessarily and increasingly non-local as the problem degenerates. This feature is also visible in our analysis, however, the numerical illustrations will show that except for extreme correlations a very small number of neighbouring points suffices. In the case of $\rho = 0.99$ and for moderate anisotropy, only 3 layers of cells in each direction are needed.

An alternative characterisation of monotone schemes via dual cones is given in [6], and this is the work most closely related to the present paper. This method of analysis is constructive as it naturally leads to the discretisation coefficients as the solution of a linear program. An efficient implementation is discussed in [5]. Where our analysis differs is that [6] formulate their approximations in terms of linear combinations of standard one-dimensional diffusion approximations in all possible directions to lattice points in a certain vicinity, while our primitive quantity is the transition probabilities directly. In so doing, we are able to condense the conditions in Section 5 of [6] into a single formula which lends itself easily to numerical illustrations and can be simplified further to an easy to evaluate rule of thumb.

The novel contributions of this paper are:

- a precise characterisation of the model parameters (variances and correlations) for which local Markov chain approximations on a lattice exist (i.e., the process moves not more than a given fixed number of nodes in every timestep), see Theorem 3.1;
- equivalently, a characterisation of monotone discretisations of (degenerate) elliptic PDEs with fixed given stencil size;
- a very simple necessary condition, see Corollary 3.2;