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A lower bound for the dispersion on the torus

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Abstract

We consider the volume of the largest axis-parallel box in the d-dimensional torus that contains no point of a given point set $\mathcal{P}_n$ with n elements. We prove that, for all natural numbers d, n and every point set $\mathcal{P}_n$, this volume is bounded from below by $\min\{1, d/n\}$. This implies the same lower bound for the discrepancy on the torus.

Keywords: Dispersion; Discrepancy; Torus

1. Introduction

The study of uniform distribution properties of n-element point sets $\mathcal{P}_n$ in the d-dimensional unit cube has attracted a lot of attention in past decades, in particular because of its strong connection to worst case errors of numerical integration using cubature rules, see e.g. [5,13,16]. There is a vast body of articles and books considering the problem of bounding the discrepancy of point sets. That is, given a probability space (X, μ) and a set $\mathcal{B}$ of measurable subsets of X, which we call *ranges*, we want to find the maximal difference between the measure of a set $B \in \mathcal{B}$ and the empirical measure induced by the finite set $\mathcal{P}_n$, i.e.

$$D(\mathcal{P}_n, \mathcal{B}) := \sup_{B\in\mathcal{B}} \left| \frac{\#(\mathcal{P}_n \cap B)}{n} - \mu(B) \right|,$$

where $\mathcal{P}_n \subset X$, $n \in \mathbb{N}$, with $\#\mathcal{P}_n = n$. In what follows we only consider $X = [0, 1]^d$, $d \geq 1$, and the Lebesgue measure μ; we write $|B| := \mu(B)$. The number $D(\mathcal{P}_n, \mathcal{B})$ is called the *discrepancy* of the point set $\mathcal{P}_n$ with respect to the ranges $\mathcal{B}$. See e.g. the monographs/surveys [4–6,13,14,16] for the state of the art, open problems and further literature on this topic.

Here, we are interested in lower bounds for this quantity that hold for every point set $\mathcal{P}_n$. In fact, we are going to bound the apparently smaller quantity

$$\mathrm{disp}(\mathcal{P}_n, \mathcal{B}) := \sup_{\substack{B\in\mathcal{B}:\\ \mathcal{P}_n\cap B=\varnothing}} |B|,$$

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which we call the *dispersion* of the point set $\mathcal{P}_n$ with respect to the ranges $\mathcal{B}$. Clearly, this is a lower bound for the discrepancy.

The notion of the dispersion was introduced by Hlawka [9] as the radius of the largest empty ball (for a given metric). In this setting there are some applications including the approximation of extreme values (Niederreiter [12]) or stochastic optimization (Yakowitz et al. [19]). The present definition was introduced by Rote and Tichy [17] together with a treatment of its value for some specific point sets and ranges. Only recently an application to the approximation of high-dimensional rank one tensors was discussed in Bachmayr et al. [3] and Novak and Rudolf [15], where the ranges are all axis-parallel boxes in $[0, 1]^d$. A polynomial-time algorithm for finding the largest empty axis-parallel box in dimension 2 was considered by Naamad, Lee and Hsu [11].

Our main interest is the complexity of the problem of finding point sets with small dispersion/discrepancy; especially the dependence on the dimension. That is, given some $\varepsilon > 0$ and $d \in \mathbb{N}$, we want to know how many points are necessary to achieve $\mathrm{disp}(\mathcal{P}_n, \mathcal{B}) \le \varepsilon$ or $D(\mathcal{P}_n, \mathcal{B}) \le \varepsilon$ for some $\mathcal{P}_n \subset [0, 1]^d$ and $\mathcal{B} \subset 2^{[0,1]^d}$. For this we define the inverse functions

$$N_0(\varepsilon, \mathcal{B}) := \min\{n\colon \mathrm{disp}(\mathcal{P}, \mathcal{B}) \le \varepsilon \text{ for some } \mathcal{P} \text{ with } \#\mathcal{P} = n\}$$

and

$$N(\varepsilon, \mathcal{B}) := \min\{n\colon D(\mathcal{P}, \mathcal{B}) \le \varepsilon \text{ for some } \mathcal{P} \text{ with } \#\mathcal{P} = n\}.$$

We have $N_0(\varepsilon, \mathcal{B}) \le N(\varepsilon, \mathcal{B})$ for every $\varepsilon, \mathcal{B}$.

For example, if $\mathcal{B} = \mathcal{B}^d_{\mathrm{ex}}$ is the set of all axis-parallel boxes contained in $[0, 1]^d$, then it is easily seen that for every point set there exists an empty box with volume larger than $1/(n+1)$; simply split the cube in $n+1$ equal parts, one of which must be empty. Moreover, it is known that with respect to the dependence on n this estimate is asymptotically optimal, i.e. there exists a sequence of point sets $(\mathcal{P}_n)_{n\in\mathbb{N}}$ such that $\mathrm{disp}(\mathcal{P}_n, \mathcal{B}^d_{\mathrm{ex}}) \le C_d/n$ for some $C_d < \infty$, see e.g. [17].[1]

However, if one considers increasing values of the dimension the situation is less clear: The best bounds to date are

$$\frac{\log_2 d}{4(n + \log_2 d)} \le \inf_{\mathcal{P}:\#\mathcal{P}=n} \mathrm{disp}(\mathcal{P}, \mathcal{B}^d_{\mathrm{ex}}) \le \frac{C^d}{n}$$

for some constant $C < \infty$, see Aistleitner et al. [2] for the lower bound and Larcher [10] for the upper bound. For a proof of a super-exponential upper bound see also Rote and Tichy [17, Prop. 3.1]. This can be rewritten as

$$(1/4 - \varepsilon)\frac{\log_2 d}{\varepsilon} \le N_0(\varepsilon, \mathcal{B}^d_{\mathrm{ex}}) \le \frac{C^d}{\varepsilon}.$$

Clearly, there is a huge difference in the behavior in d for the upper and the lower bound.

If we consider the discrepancy instead, then even the order in ε^{-1} differs in the upper and the lower bounds, i.e. for small enough $\varepsilon \le \varepsilon_0$ and all $d \in \mathbb{N}$ we have

$$c\, d\, \varepsilon^{-1} \le N(\varepsilon, \mathcal{B}^d_{\mathrm{ex}}) \le C\, d\, \varepsilon^{-2}$$

with some constants $0 < c, C < \infty$.[2] The lower bound is due to Hinrichs [8] and the upper bound was proven by Heinrich et al. [7]. To narrow the gap in the ε-behavior while keeping a polynomial behavior in d is a long-standing open problem, see also Novak and Woźniakowski [16] for more results/problems in this area.

Nevertheless, for fixed, small $\varepsilon > 0$ the d-dependence of $N(\varepsilon, \mathcal{B}^d_{\mathrm{ex}})$ is known to be linear. This motivates us to study the same problem for the dispersion. Unfortunately, we were not able to solve this problem for the ranges $\mathcal{B}^d_{\mathrm{ex}}$. Instead, we consider the "periodic" version of this problem.

More precisely, we regard the unit cube as the torus and consider the *periodic ranges* $\mathcal{B}^d_{\mathrm{per}}$ that are defined by

$$\mathcal{B}^d_{\mathrm{per}} := \left\{B_1(x, y)\colon x, y \in [0, 1]^d\right\}, \tag{1}$$

[1] Note that for the discrepancy such an inequality cannot hold for any sequence of point sets, see Roth [18].

[2] If one considers only boxes that are anchored at the origin, i.e. the star-discrepancy, then one can choose $c = \varepsilon_0 = 1/(32e^2) \approx 0.00423$ [8] and $C = 100$ [1].