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Weak convergence of Galerkin approximations of stochastic partial differential equations driven by additive Lévy noise

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Abstract

This work considers weak approximations of stochastic partial differential equations (SPDEs) driven by Lévy noise. The SPDEs at hand are parabolic with additive noise processes. A weak-convergence rate for the corresponding Galerkin Finite Element approximation is derived. The convergence result is derived by use of the Malliavin derivative rather than the common approach via the Kolmogorov backward equation.

Keywords: Weak convergence; Stochastic partial differential equation; Lévy noise; Malliavin calculus

1. Introduction

In contrast to partial differential equations, the error analysis of approximations of solutions to stochastic (partial) differential equations (SPDEs) allows for two conceptually different approaches: weak and strong. Both of these kinds of error analysis for SPDEs have been actively researched during the last two decades. While the strong (or pathwise) error has been the subject of a vast array of publications, the weak error, which is computed in terms of moments of the solution process, has, to the date, garnered considerably less attention.

In this paper we consider weak-convergence rates of Galerkin Finite Element approximations of solutions to the parabolic stochastic partial differential equation given, for $t \in (0, T]$, $T < +\infty$, by

$$\begin{aligned} dX(t) + AX(t)dt &= f(t)dt + G(t)dL(t), \\ X(0) &= x_0 \in H. \end{aligned} \tag{1}$$

By H we denote a separable Hilbert space, A is a linear operator on H, f maps $\mathbb{T} := [0, T]$ into H and G is a mapping from $\mathbb{T}$ into the linear, bounded operators form some separable Hilbert space U (not necessarily equal to H) into H. Further, L denotes a Lévy process defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and it takes values in U.

For solutions to SPDEs like Eq. (1), the strong-error rate of Galerkin approximations has been considered, among others, in [3,4,6,7,9,13–15,17,19,20,23]. In these references, SPDEs driven by either Gaussian or Lévy noises are

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treated. Publications on weak approximations and their error analysis are, among others, [1,2,5,8,10,12,16,18,21,22], where, to a great extent, SPDEs driven by Gaussian processes are considered.

In this paper, we consider an equivalent approach as in [22] and combine it with the recent results on Malliavin calculus for Lévy driven SPDEs in [11]. In our main result, Theorem 3.4, we show that the weak-convergence rate is essentially twice the strong-convergence rate. This is akin to the findings in [21, Theorem 4.5], where a similar equation is treated and a weak-convergence result for a Galerkin approximation via the backward Kolmogorov equation is derived. Our methodology, however, differs considerably and, with it, the regularity assumptions required on the functional of the solution.

The paper is organized as follows: In Section 2 we provide the notation and the results on Malliavin calculus for infinite dimensional Lévy processes required for the weak-convergence result. In Section 3, we introduce the stochastic partial differential equation in question, as well as its approximation. We then proceed with the proof of the main result on weak convergence of this approximation.

2. Notation and preliminaries

Let $(U, \langle\cdot,\cdot\rangle_U)$ and $(H, \langle\cdot,\cdot\rangle_H)$ be separable Hilbert spaces and let $L(U;H)$ be the space of all linear bounded operators from U into H endowed with the usual supremum norm. If $U = H$, the abbreviation $L(U) := L(U;U)$ is used. An element $G \in L(U;H)$ is said to be a nuclear operator if there exists a sequence $(x_k, k \in \mathbb{N})$ in H and a sequence $(y_k, k \in \mathbb{N})$ in U such that

$$\sum_{k\in\mathbb{N}} \|x_k\|_H \, \|y_k\|_U < +\infty$$

and G has, for $z \in U$ the representation

$$Gz = \sum_{k\in\mathbb{N}} \langle z, y_k\rangle_U \, x_k.$$

The space of all nuclear operators from U into H, endowed with the norm

$$\|G\|_{L_N(U;H)} := \inf\left\{\sum_{k\in\mathbb{N}} \|x_k\|_H \, \|y_k\|_U \;\middle|\; Gz = \sum_{k=1}^{\infty} \langle z, y_k\rangle_U \, x_k\right\}$$

is a Banach space, and is denoted by $L_N(U;H)$. If $U = H$, we use the abbreviation $L_N(U)$. Furthermore, let $L_N^+(U)$ denote the space of all nonnegative, symmetric, nuclear operators on U, i.e.,

$$L_N^+(U) := \left\{G \in L_N(U) | \, \langle Gy, y\rangle_U \geq 0, \; \langle Gy, z\rangle_U = \langle y, Gz\rangle_U \; \text{ for all } y, z \in U\right\}.$$

An operator $G \in L(U;H)$ is called a Hilbert–Schmidt operator if

$$\|G\|^2_{L_{HS}(U;H)} := \sum_{k=1}^{\infty} \|Ge_k\|_H^2 < +\infty$$

for any orthonormal basis $(e_k, k \in \mathbb{N})$ of U. The space of all Hilbert–Schmidt operators $(L_{HS}(U;H), \|\cdot\|_{L_{HS}(U;H)})$ is a Hilbert space with inner product given by

$$\left\langle G, \tilde{G}\right\rangle_{L_{HS}(U;H)} := \sum_{k=1}^{\infty} \left\langle Ge_k, \tilde{G}e_k\right\rangle_H,$$

for $G, \; \tilde{G} \in L_{HS}(U;H)$ and any orthonormal basis $(e_k, k \in \mathbb{N})$ of U. If $U = H$, the abbreviation $L_{HS}(U) := L_{HS}(U;U)$ is used.

Given a measure space $(S, \mathcal{S}, \mu)$ and $r \in [1, +\infty)$, we denote by $L^r(S;H)$ the space of all $\mathcal{S}$-$\mathcal{B}(H)$-measurable mappings $f : S \to H$ with finite norm

$$\|f\|_{L^r(S;H)} := \left(\int_S \|f\|_H^r \, d\mu\right)^{\frac{1}{r}},$$

where $\mathcal{B}(H)$ denotes the Borel σ-algebra over H.