Engineering Analysis with Boundary Elements 37 (2013) 977–989

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Numerical study of the three-dimensional wave equation using the mesh-free kp-Ritz method

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ARTICLE INFO

Article history:
Received 19 September 2012
Accepted 4 April 2013
Available online 13 May 2013

Keywords:
Mesh-free kp-Ritz method
Meshless method
Wave equation
Error estimate
Convergence analysis

ABSTRACT

This paper deals with numerical modeling of three-dimensional linear wave propagation based on the mesh-free kp-Ritz method. The mesh-free kernel particle estimate is employed to approximate the 3D displacement field. A system of discrete equations is obtained through application of the Ritz minimization procedure to the energy expressions. Convergence analysis and error estimates of the kp-Ritz method for three-dimensional wave equation are also presented in the paper. From the error analysis, we found that the error bound between the numerical and the exact solution is directly related to the radii of weight functions and the time step length. Effectiveness of the kp-Ritz method for three-dimensional wave equation is investigated by three numerical examples.

1. Introduction

The finite element method (FEM) and boundary element method (BEM) are well established numerical techniques which have been applied to obtain numerical solutions for various problems in the field of engineering and science. Even though these methods are very effective for solving various kinds of partial differential equations, they have their own limitations also [1–3]. For FEM, the need to produce a body-fitted mesh in two- and three-dimensional problems makes this method time-consuming and difficult to use. The BEM requires a domain node distribution for an inhomogeneous equation, in addition to a boundary mesh, and depends on the fundamental solutions. Meshless techniques have attracted the attention of researchers since they help to avoid these problems. In a meshless (mesh-free) method, a set of scattered nodes are used, instead of meshing of the domain of the problem. For many engineering problems, such as large deformation and crack growth, use of the re-meshing technique is necessary to deal with extremely large deformations or fractures of the mesh. The meshless method is a new and interesting numerical technique that can solve many engineering problems which are not amenable to be solved with conventional numerical methods, with minimum or no meshing at all [4].

Some meshless methods have been developed, such as smooth particle hydrodynamics methods (SPH) [5], radial basis function (RBF) [6], element free Galerkin method (EFG) [7], reproducing kernel particle method (RKPM) [8–10], meshless local Petrov–Galerkin method (MLPG)[11] and the finite point method(FPM) [12] and so on.

The wave equations model vibrations of structures (e.g., buildings, beams and machines) and are the basis for fundamental equations of atomic physics. The wave equation usually describes water waves, vibrations of a string, propagation of sound waves and transmission of electric signals in a cable, etc. The typical model that describes the wave equation is an initial boundary value problem valid in a bounded domain or an initial value problem valid in an unbounded domain. It is interesting to note that two initial conditions need to be prescribed, namely the initial displacement $u(x,y,z,0)=\varphi_1(x,y,z)$ and the initial velocity $u_t(x,y,z,0)=\varphi_2(x,y,z)$, both at starting time $t=0$. Unlike the heat equation, the wave equation contains the term u_{tt} which represents the vertical acceleration at point (x,y,z). The wave equation plays a significant role in various physical problems and is required in diverse areas of science and engineering [13–16]. The wave equations arise in many kinds of engineering problems and have attracted much research attention recently. Second- and fourth-order accuracy operator splitting methods for the three-dimensional linear hyperbolic equations have been discussed [17–19]. These schemes are conditionally stable. Mohanty et al. [20–22] proposed unconditionally stable difference schemes for the linear second-order hyperbolic equation with first-order time derivative terms in one, two and three space dimensions. Mehdi Dehghan [23–27] has done much work on wave equations based on finite difference method and RBF.

The Ritz [28] approximation approach, developed almost a century ago, is a generalization of the Rayleigh [29] method. It is

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http://dx.doi.org/10.1016/j.enganabound.2013.04.001

based on the principle that a resonant vibrating system completely interchanges its kinetic and potential energy forms. In the Rayleigh method, a single trial function for the mode shape satisfying at least the geometric boundary conditions is employed, and then by equating the maximum kinetic and potential energies, an upper bound frequency solution is obtained. The Ritz or Rayleigh–Ritz method is a proven approximation technique in computational mechanics; notable works include Kitipornchai et al. [30], Liew et al. [31,32], Cheung and Zhou [33,34], Zeng and Bert [35], Su and Xiang [36], Lim and Liew [37] and Liew and Feng [38].

This paper employs the reproducing kernel particle estimation to study the three-dimensional wave equation. In this study, the displacement field is approximated as these kernel functions, an energy formulation is formulated and a system of discrete equations is obtained by the Ritz minimization procedure. In the conventional Ritz method, it is difficult to choose appropriate functions to satisfy some complicated boundary conditions; eigenvalue equations need to be reevaluated for different boundary conditions. In the present kp-Ritz method, a standard weight function is employed to express the interior field and boundary conditions are enforced by the penalty method, thereby avoiding disadvantages of the conventional Ritz method. Error estimates and convergence analysis of this method for three-dimensional wave equation are also discussed. A few selected example problems are used to study the applicability of the method and the numerical results are compared with the existing exact solutions.

2. kp-Ritz method for the three-dimensional wave equation

2.1. Energy formulation

Consider the following three-dimensional linear wave equation of the form:

$$\frac{\partial^2 u}{\partial t^2}+2a\frac{\partial u}{\partial t}+b^2u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+f(x,y,z,t),$$
$$0\le x\le a_1,\quad 0\le y\le b_1,\quad 0\le z\le c_1,\quad t>0 \tag{1a}$$

with initial conditions

$$u(x,y,z,0)=\varphi_1(x,y,z) \tag{1b}$$

$$\frac{\partial u(x,y,z,0)}{\partial t}=\varphi_2(x,y,z) \tag{1c}$$

and boundary conditions

$$u(0,y,z,t)=f_0(y,z,t),\quad u(a_1,y,z,t)=f_1(y,z,t) \tag{1d}$$

$$u(x,0,z,t)=g_0(x,z,t),\quad u(x,b_1,z,t)=g_1(x,z,t) \tag{1e}$$

$$u(x,y,0,t)=h_0(x,y,t),\quad u(x,y,c_1,t)=h_1(x,y,t) \tag{1f}$$

The above PDE is an important hyperbolic equation that represents three-space dimensional damped wave equation with a source term, also called the telegraph equation. Eq. (1) models the mixture of diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to standard heat or mass transport equation. It is commonly used in signal analysis for transmission and propagation of electrical signals and also has applications in other fields. Numerical solution of the damped wave equation is of great importance in wave phenomena.

The weighted integral form of Eq. (1a) is obtained as follows:

$$\int_V w\cdot\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+f(x,y,t)-\frac{\partial^2 u}{\partial t^2}-2a\frac{\partial u}{\partial t}-b^2u\right]dV=0 \tag{2}$$

The weak form of Eq. (2) is

$$\int_V\left[\nabla^T w\nabla u-wf+w\frac{\partial^2 u}{\partial t^2}+2aw\frac{\partial u}{\partial t}+b^2wu\right]dV$$
$$-\int_S w\left(\frac{\partial u}{\partial x}n_x+\frac{\partial u}{\partial x}n_y+\frac{\partial u}{\partial z}n_z\right)dS=0 \tag{3}$$

The energy functional $\Pi(u)$ can be written as

$$\Pi(u)=\frac{1}{2}\int_\Omega \nabla^T u\cdot\nabla u\,d\Omega+\int_V\left[u\frac{\partial^2 u}{\partial t^2}+2au\frac{\partial u}{\partial t}+b^2u^2-uf\right]dV \tag{4}$$

2.2. Three-dimensional kernel particle shape functions

Approximation of the displacement field can be expressed as

$$u(x,t)=\sum_{I=1}^{n}\Phi_I(x)\cdot U_I(t)=\Phi(x)\cdot U \tag{5}$$

where n is the total number of particles, $\Phi_I(x)$ are the shape functions and $U_I(t)$ are unknown nodal values of u at a sampling point I. Based on the reproducing kernel particle estimate, the shape function is given by

$$\Phi_I(\boldsymbol{x})=C(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x_I})w_r(\boldsymbol{x}-\boldsymbol{x_I}) \tag{6}$$

where $C(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x_I})$ is the correcting function and $w_r(\boldsymbol{x}-\boldsymbol{x_I})$ is the kernel function. The kernel function is expressed as

$$w_r(\boldsymbol{x}-\boldsymbol{x}_I)=\frac{1}{r}\phi\left(\frac{\boldsymbol{x}-\boldsymbol{x}_I}{r}\right) \tag{7}$$

and the correction function is written as

$$c(\boldsymbol{x};\boldsymbol{x}-\boldsymbol{x}')=\sum_{i=1}^{m}p_i(\boldsymbol{x}-\boldsymbol{x}')\cdot b_i(\boldsymbol{x})=p^T(\boldsymbol{x}-\boldsymbol{x}')b(\boldsymbol{x}),\quad(\boldsymbol{x}\in\Omega) \tag{8}$$

where

$$\boldsymbol{P}=\begin{bmatrix}p_1(\boldsymbol{x}-\boldsymbol{x}_1) & p_1(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_1(\boldsymbol{x}-\boldsymbol{x}_n)\\ p_2(\boldsymbol{x}-\boldsymbol{x}_1) & p_2(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_2(\boldsymbol{x}-\boldsymbol{x}_n)\\ \vdots & \vdots & \ddots & \vdots\\ p_m(\boldsymbol{x}-\boldsymbol{x}_1) & p_m(\boldsymbol{x}-\boldsymbol{x}_2) & \dots & p_m(\boldsymbol{x}-\boldsymbol{x}_n)\end{bmatrix} \tag{9}$$

$$b^T(x)=(b_1(x),b_2(x),\dots,b_m(x)) \tag{10}$$

where m is the number of terms in the basis, $p_i(\boldsymbol{x}-\boldsymbol{x}')$ are monomial basis functions and $b_i(\mathbf{x})$ are coefficients of monomial basis functions to be determined. Generally, the basis can be

Linear basis ($p=1$):

$$\boldsymbol{p}^T=(1,x_1-x_1',\,x_2-x_2'),\quad(2D) \tag{11}$$

$$\boldsymbol{p}^T=(1,x_1-x_1',\,x_2-x_2',\,x_3-x_3'),\quad(3D) \tag{12}$$

Quadratic basis ($p=2$):

$$\boldsymbol{p}^T=(1,x_1-x'_1,x_2-x'_2,(x_1-x'_1)^2,(x_1-x'_1)(x_2-x'_2),(x_2-x'_2)^2),\quad(2D) \tag{13}$$

$$\boldsymbol{p}^T=(1,x_1-x_1',\,x_2-x_2',\,x_3-x_3',(x_1-x_1')^2,(x_2-x_2')^2,(x_3-x_3')^2,$$
$$(x_1-x_1')(x_2-x_2'),(x_1-x_1')(x_3-x_3'),(x_2-x_2')(x_3-x_3')),\quad(3D) \tag{14}$$

Thus the shape function can be written as

$$\Phi_I(\boldsymbol{x})=b^T(\boldsymbol{x})P(\boldsymbol{x}-\boldsymbol{x_I})w_r(\boldsymbol{x}-\boldsymbol{x_I}) \tag{15}$$

Eq. (15) can be rewritten as

$$\Phi_I(\boldsymbol{x})=b^T(\boldsymbol{x})B_I(\boldsymbol{x}-\boldsymbol{x_I}) \tag{16}$$

where

$$B_I(\boldsymbol{x}-\boldsymbol{x_I})=P(\boldsymbol{x}-\boldsymbol{x_I})\phi_a(\boldsymbol{x}-\boldsymbol{x}_I) \tag{17}$$

$$\boldsymbol{b}(\boldsymbol{x})=M^{-1}(\boldsymbol{x})\cdot H \tag{18}$$

$$H=(1,0,\dots,0)^T \tag{19}$$