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Error estimates for the full discretization of a semilinear parabolic problem with an unknown source

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Abstract

This paper is devoted to the study of an inverse semilinear parabolic problem. The problem contains an unknown solely time-dependent source function p and a homogeneous Dirichlet boundary condition. Moreover, an integral measurement of the total energy/mass in the domain is given. A full-discrete finite element scheme to approximate the unique weak solution is designed. For the time discretization backward Euler's method is used. For the space discretization the finite element method is applied. Various error estimates are derived, depending on the regularity of the data and on the choice of the finite elements.
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1. Introduction

Over the past few years, many papers have been published concerning inverse problems (IPs) with an unknown source function. This unknown source function can depend either only on the space variable, cf. [8,19,24,28], or only the time variable, cf. [3,6,7,10,11,15,17,21,23,25,27,26,29–31], alternatively on both, cf. [5,13,14,18,16]. In this paper, we focus on an inverse source problem of the second type. To reconstruct a solely time-dependent source a supplementary time-dependent measurement is needed. In [10,21,23,25,29,30], this supplementary measurement is a temperature measurement at an internal point of the spatial domain. In [15] the temperature has been measured at a boundary point of the spatial domain. Nevertheless, the measurement is not always local: in [3] the authors measured the temperature along a curve inside the spatial domain. Also, an integral overdetermination is frequently used in various IPs for evolutionary equations, cf. [17,22,27,26,31] and the references therein. A physical motivation for the use of integral measurements can be found in [22, p. 378].

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In this contribution, we study an inverse semilinear parabolic source problem with zero Dirichlet boundary condition (BC). More specifically, this problem looks like

$$\begin{cases} \partial_t u(t,\boldsymbol{x}) - \Delta u(t,\boldsymbol{x}) &= p(t)f(\boldsymbol{x}) + g(u(t,\boldsymbol{x})) + r(t,\boldsymbol{x}), & (t,\boldsymbol{x}) \in (0,T] \times \Omega, \\ u(t,\boldsymbol{x}) &= 0, & (t,\boldsymbol{x}) \in (0,T] \times \Gamma, \\ u(0,\boldsymbol{x}) &= u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ \int_\Omega u(t,\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= m(t), & t \in [0,T], \end{cases} \tag{1}$$

with $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N} = \{1, 2, 3, \ldots\}$, a bounded domain with a Lipschitz continuous boundary Γ. The data functions f, g, r, u_0 and the measurement m are supposed to be known. However, the source parameter $p(t)$ and the function $u(t, \boldsymbol{x})$ need to be reconstructed considering that the total energy/mass in the spatial domain equals the integral overdetermination $m(t)$. Throughout this paper we assume that g is a Lipschitz continuous function, $f \in L_2(\Omega)$, $\int_\Omega f \neq 0$, $r : [0, T] \to L_2(\Omega)$ and $m : [0, T] \to \mathbb{R}$. Moreover, we use $\boldsymbol{n}$ to denote the outward unit vector on Γ.

Problem (1) has applications e.g. in heat conduction theory and in reaction–diffusion processes when looking for respectively the heat intensity and the concentration p of a source at every time point $t \in [0, T]$. The function f then simply gives the spatial position of the source or it describes how the intensity or concentration of the source depends on the spatial position.

A literature study has given us some local, cf. [31], and global, cf. [3,6,10,11,17,27,26], existence and uniqueness results for inverse problems with an unknown time-dependent source function, as well as some time-discrete numerical procedures to solve these problems, cf. [6,11,27,26]. In particular, the global well-posedness of problem (1) has been addressed in [11], where also a constructive and convergent time-discrete numerical scheme has been designed using Rothe's Method. Yet, only weak convergence in $L_2(0, T)$ of the numerical approximations of the source function p has been shown. In [6], the same method has been applied to a similar problem in which the unknown time-dependent heat source also appears as convolution kernel. However, the techniques used in [6] will not work for a Dirichlet problem. Error estimates for the time discretization in [6] have been derived in [7]. Nevertheless, there is a lack of information about full discretization methods and the corresponding error estimates for inverse source problems like problem (1). We are only aware of a few full-discrete numerical procedures to determine the solution of some specific inverse source problems in one dimension. Some of them are based on finite difference methods, cf. [30]. Others depend on the Tikhonov regularization method combined with either the boundary element method, cf. [17,10], or the method of fundamental solutions, cf. [29]. Yet, we have not found any results for the use of the finite element method to discretize a problem like (1) in space, even not in a one-dimensional setting.

The first goal of this paper is to present a full-discrete numerical scheme, based on backward Euler's method for the time discretization and on the finite element method for the space discretization, to solve problem (1). The second goal of this paper is to derive some corresponding error estimates which imply the strong convergence of the numerical approximations towards the exact solution in some appropriate spaces.

The outline of the paper is as follows: first, we repeat the most important results from [11] in Section 2. Next, in Section 3, a full-discrete numerical scheme to approximate the solution to problem (1) is proposed and a priori estimates are derived. Section 4 investigates how various error estimates are proved, depending on the regularity of the data and on the choice of the finite elements for the space discretization. In Section 5, a numerical experiment in one dimension supports the theoretically obtained convergence rates in the case that Hermite Finite Elements are applied. Finally, a conclusion is stated in Section 6.

Remark 1. We consider the values C, ε and C_ε to be generic and positive constants (independent of the discretization parameters), where ε is arbitrarily small and C_ε arbitrarily large, i.e. $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$. We will use the same notations for different constants, but the meaning will be clear from the context.

2. Time-discrete scheme

The existence of a unique solution $\{u, p\} \in \left(C\left([0, T], H_0^1(\Omega)\right) \cap L_\infty((0, T), V)\right) \times L_2(0, T)$ to problem (1) has been proved in [11]. The variational framework has been designed as follows. First, the partial differential equation (PDE) in (1) has been integrated over the spatial domain to obtain an expression for the unknown function p in terms

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