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A generalized Taylor method of order three for the solution of initial value problems in standard and infinity floating-point arithmetic[☆]

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Abstract

A well-known drawback of algorithms based on Taylor series formulae is that the explicit calculation of higher order derivatives formally is an over-elaborate task. To avoid the analytical computation of the successive derivatives, numeric and automatic differentiation are usually used. A recent alternative to these techniques is based on the calculation of higher derivatives by using the Infinity Computer—a new computational device allowing one to work numerically with infinities and infinitesimals. Two variants of a one-step multi-point method closely related to the classical Taylor formula of order three are considered. It is shown that the new formula is order three accurate, though requiring only the first two derivatives of $y(t)$ (rather than three if compared with the corresponding Taylor formula of order three). To get numerical evidence of the theoretical results, a few test problems are solved by means of the new methods and the obtained results are compared with the performance of Taylor methods of order up to four.
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1. Introduction

The Taylor series method is one of the earliest algorithms to approximate the solution of initial value problems

$$\begin{cases} y' = f(t, y), \quad t \in [t_0, T], \\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed sufficiently differentiable.

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Newton and Euler describe this approach in their seminal works of the 18th century. Since then, many authors mention Taylor series methods and some codes have been developed for both ODEs and DAEs: within the recent literature, we mention [3–5,15,53,54].

A well-known drawback of the algorithms based on Taylor series formulae is that the explicit calculation of higher order derivatives formally is an over-elaborate task, especially when the dimension n of the system is not small. To avoid the analytical computation of the successive partial derivatives involved in the truncated Taylor expansion of f, a numerical differentiation approach has been often considered (see, for example, [24,25]). A further interest in Taylor series methods stemmed from considering automatic rather than numerical differentiation, which makes use of specific tools based on the involved elementary functions (see [29]) and allows for a speed up of the overall computation.

We report two instances for which the use of Taylor series methods has proved to be a powerful tool:

- The analysis of the stability properties of equilibria and periodic orbits of dynamical systems often requires an accurate integration of the variational equations. Within this context, high-order Taylor series methods have been successfully exploited to correctly reproduce the highly oscillatory behavior of their solutions, avoiding extremely small stepsizes during the integration procedure. In most cases the variational equations are slight modifications of the original ones, so that it is possible to formulate the integration algorithm for both systems with little added effort.
- In some physical problems [1,2,22] it is important to approximate the solution with a very high precision, as in the determination of normal forms of differential systems, initial conditions for periodic problems, numerical detection of periodic orbits, computation of physical constants, etc. The Taylor method, just by increasing the degree of the formulae, permits high-precision integration, provided a multi-precision library is also used.

A recent alternative to numeric and automatic differentiation is based on the calculation of higher derivatives by using the Infinity Computer which is equipped with a new numeral system (see [30,33,38,32,42,47]) for performing numerical computations with infinite and infinitesimal quantities. The possibility to work with numerical infinitesimals allows one both to calculate the exact values of the derivatives numerically without finding the respective derivatives analytically and to work with infinitesimal stepsizes. The first attempts to use the Infinity Computer in this direction have been done in [39,43,45].

In order to see the place of the new approach in the historical panorama of ideas dealing with infinite and infinitesimal, see [17–19,26,35,37,49]. In particular, connections of the new approach with bijections are studied in [19] and metamathematical investigations on the theory and its non-contradictory identification can be found in [18]. The new methodology has been successfully used in such fields as numerical differentiation and optimization (see [9,39,56]), fractals (see [13,14,31,34,41,48]), models for percolation and biological processes (see [13,14,52,41]), hyperbolic geometry (see [20,21]), infinite series (see [16,35,40,55]), set theory, lexicographic ordering, and Turing machines (see [37,46,44,49,50]), cellular automata (see [10–12]), etc.

The paper is structured as follows. Section 2 gives a brief introduction into the work with numerical infinitesimals and infinities on the Infinity Computer. Section 3 introduces the new methods. Convergence and stability analysis of the new algorithms is performed in Section 4. Some numerical illustrations are provided in Section 5. Finally, Section 6 concludes the paper.

We stress that the action played by the Infinity Computer only concerns the accurate evaluation of the derivatives appearing in the expression of the methods. As a matter of fact, the variant of the standard Taylor methods we are going to introduce may be efficiently implemented in both standard and infinity floating-point arithmetic.

2. Numerical infinitesimals and infinities

In our everyday activities with finite numbers the *same* finite numerals[1] are used for *different* purposes (e.g., the same numeral 9 can be used to express the number of elements of a set, to indicate the position of an element in a sequence, and to execute practical computations). In contrast, when we face the necessity to work with infinities or infinitesimals, the situation changes drastically. In fact, in this case *different* numerals are used to work with infinities

[1] There exists an important distinction between *numbers* and *numerals*. A *numeral* is a symbol (or a group of symbols) that represents a *number*. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols ‘9’, ‘nine’, ‘IIIIIIIII’, and‘IX’, are different numerals, but they all represent the same number.

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