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A dual-mixed finite element method for quasi-Newtonian flows whose viscosity obeys a power law or the Carreau law

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Abstract

The aim of this work is a construction of a dual mixed finite element method for a quasi-Newtonian flow obeying the Carreau or power law. This method is based on the introduction of the stress tensor as a new variable and the reformulation of the governing equations as a twofold saddle point problem. The derived formulation possesses local (i.e. at element level) conservation properties (conservation of the momentum and the mass) as for finite volume methods. Based on such a formulation, a mixed finite element is constructed and analyzed. We prove that the continuous problem and its approximation are well posed, and derive error estimates.

Keywords: Mixed finite element; Quasi-Newtonian; Power law; Carreau law

1. Introduction

We propose a dual-mixed formulation for non-Newtonian fluid flow where the fluid viscosity is assumed to be nonlinear function of the rate of strain tensor. The governing equations arise in modeling flows of, for example, biological fluids, lubricants, paints and polymeric fluids. In [9,10], we have introduced and analyzed a dual-mixed finite element method for quasi-Newtonian fluid flow obeying to the power law. *A priori* error estimates for the finite element approximation were proved in [9], while *a posteriori* error estimation was provided in [10]. However, in both [9,10] the analysis used the assumption that the equation describing the stress tensor in terms of the rate of strain tensor was invertible to give the rate of strain tensor as a function of the stress tensor. The mixed finite element method developed in [9] possesses local (i.e., at element level) conservation properties (conservation of the momentum and the mass) as in the finite volume methods. Furthermore, it allows the approximations of all the physical variables (stress, velocity and pressure).

The aim of this work is to extend our investigations by avoiding the assumption of expressing the rate of strain tensor as a function of the stress tensor. As example of such a situation is a non-Newtonian fluid flow obeying

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the Carreau law. For this purpose, we introduce an additional variable for the rate of strain tensor and reformulate the governing equations as a twofold saddle point problem. It must be noted that this kind of approach has been introduced and analyzed in [11,7] for a class of quasi-Newtonian Stokes flows. However, in both [11,7] the tensor gradient of the velocity was used instead of the rate of strain tensor. The fact to use the rate of strain tensor introduces a major difficulty in the construction of mixed finite element methods (for more details, see [4]). The difficulty lies essentially in the symmetry of this tensor. One way to overcome this difficulty is to relax the symmetry of this tensor by a Lagrange multiplier. We will present our dual-mixed formulation and establish well posedness. The mixed finite element for this formulation will be provided and the associated a priori error estimates are then derived. The error estimates are optimal and are the same as the ones obtained in the particular case of Power law [9].

The outline of the paper is as follows: In Section 2, the governing equations and the mixed formulation of non Newtonian flows whose viscosity obeys a power law or the Carreau law are presented. Existence and uniqueness results are given in Section 3. In Section 4, we introduce our finite element approximation and establish well posedness. The *a priori* error estimates are then derived and the last section is devoted to the conclusion.

2. Governing equations and mixed formulation

Governed by the classical Stokes problem, the Newtonian fluid flows are a reasonable approximation of the more realistic non-Newtonian fluids (quasi-Newtonian or Viscoelastic). In the case of quasi-Newtonian fluids, the viscosity is a function of gradient tensor, temperature, time, etc. For a steady and creeping flow of an incompressible quasi-Newtonian fluid, the most used formulation is based on the strain rate tensor.

In that case, for Ω a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ and a given mass forces $\boldsymbol{f}$ defined in Ω, the combination of the conservation equations leads to the following Nonlinear Stokes problem:

$$\begin{cases} -\boldsymbol{div}\Big(2\nu(|\boldsymbol{d}(\boldsymbol{u})|)\boldsymbol{d}(\boldsymbol{u})\Big) + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ div\, \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $\boldsymbol{u}$ and p, the unknowns of the problem, are the velocity and pressure, respectively. $\boldsymbol{d}(\boldsymbol{u}) = \frac{1}{2}\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t\big)$ is the strain rate tensor, and $|\boldsymbol{d}(\boldsymbol{u})|^2 = \sum_{i,j=1}^{2} \boldsymbol{d}(\boldsymbol{u})_{ij}^2$.

For $\nu_0 > 0$ a reference viscosity and r a fluid characteristic real parameter verifying $1 < r < \infty$, the viscosity function $\nu(\cdot)$, depending on $|\boldsymbol{d}(\boldsymbol{u})|$, is usually given by one of the two following famous models:

$$\nu(x) = \nu_0 x^{r-2}, \quad \forall x \in \mathbb{R}_+, \text{ for the Power law model, and}$$
$$\nu(x) = \nu_0 \quad \left(1 + x^2\right)^{(r-2)/2}, \quad \forall x \in \mathbb{R}_+, \text{ for the Carreau model.}$$

Finally, system (1) is supplemented by a set of boundary conditions.

Remark 2.1. For $r = 2$, both models provide the well known classical Stokes problem:

$$\begin{cases} -2\nu_0\, \boldsymbol{div}(\boldsymbol{d}(\boldsymbol{u})) + \nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ div\, \boldsymbol{u} & = & 0 & \text{in } \Omega, \end{cases}$$

corresponding to a Newtonian fluid flow.

The generalized Stokes problem (1) and its approximation by standard finite elements was first studied in Baranger and Najib [1]. Extensions and improvements of the error bounds have been obtained in Sandri [17] and Barrett and Liu [2,3].

In these works, only the primal variables velocity and pressure are taken into account. But, for various reasons, we need information on other variables such as velocity gradients $\nabla\boldsymbol{u}$, strain rate tensor $\boldsymbol{d}(\boldsymbol{u})$, and extra-stress tensor $\boldsymbol{\sigma} = 2\nu(|\boldsymbol{d}(\boldsymbol{u})|)\boldsymbol{d}(\boldsymbol{u})$. In that case, it is necessary to build appropriate mixed formulations.

On the other hand, in connection with the use of the gradient tensor $\nabla\boldsymbol{u}$ which corresponds to the Ladyzhenskaya model [14]:

$$\nu(|\nabla\boldsymbol{u}|) = (\nu_0 + \nu_1\, |\nabla\boldsymbol{u}|)^{r-2}, \quad \nu_0 \geq 0,\ \nu_1 > 0,\ r > 1,$$