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Towards a unified multiresolution scheme for treating discontinuities in differential equations with uncertainties

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Abstract

In the present work, a method for solving partial differential equations with uncertainties is presented. A multiresolution method, permitting to compute statistics for the entire solution and in presence of a whatever form of the probability density function, is extended to perform an adaptation in both physical and stochastic spaces. The efficiency of this strategy, in terms of refinement/coarsening capabilities, is demonstrated on several test-cases by comparing with respect to other more classical techniques, namely Monte Carlo (MC) and Polynomial Chaos (PC). Finally, the proposed strategy is applied to the heat equation showing very promising results in terms of accuracy, convergence and regularity.

Keywords: Multiresolution; Differential equations; Uncertainty quantification; Finite element; Heat equation

1. Introduction

In the last fifty years a strong effort has been devoted to develop efficient numerical methods for solving partial differential equations. However, estimating the quality of the prediction of a numerical simulation remains still challenging. One issue is related to the uncertainties affecting the physical model and/or the initial/boundary conditions. A general consensus exists about the need to take into account experimental and modeling uncertainty in the numerical simulations. Uncertainty Quantification (UQ) is a branch of the numerical analysis focused on the uncertainty assessment and the estimation of a confidence interval for the quantity of interest.

A classical UQ method is the Monte Carlo (MC) method [36]. This method is the most common in the UQ community due to its flexibility. The rate of convergence results to be very slow $\mathcal{O}(1/\sqrt{N})$ [15], but the advantage is its independence from the number of stochastic dimensions. Over the last years, many variants of the method have been proposed to accelerate its convergence [41,15,26], though they cannot tackle its prohibitive cost. Following the seminal work of Giles in [25] about stochastic ordinary differential equations, recently a great interest has been devoted to the

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so-called Multi Level Monte Carlo (MLMC) techniques for stochastic pdes. For instance in [16], the MLMC technique is introduced for elliptic pdes with random coefficients. The idea of the MLMC is quite simple and consists in sampling the quantity of interest over different spatial mesh with the aim to reduce the computational cost with respect to the classic MC for a prescribed overall (spatial–stochastic) accuracy. Another class of methods is the so-called Stochastic Collocation (SC) approach. Two different approaches are commonly employed in literature [46]: the first is based on the Lagrange interpolation technique, while the second one relies on pseudo-spectral expansion (this second technique in this context is equivalent to the non-intrusive Polynomial Chaos described in the following). In both cases, the first step is to recover the coefficients of the expansion, *i.e.* of the Lagrange polynomial basis or the spectral basis. Once the continuous representation of the function is obtained, the statistics can be obtained directly by using some quadrature techniques. The method is conceptually straightforward, however the selection of nodes is a nontrivial problem in multiple dimensional spaces [46]. Therefore, it has been proposed, e.g. in [10,47], to use as collocation points the zeros of suitable tensor product orthogonal polynomials. Note that these methods are often coupled to sparse grid techniques for high-dimensional problems, in order to reduce the computational cost. The sparse grid strategy has been proposed by Smolyak [40]. It consists in the interpolation of a function in a reduced subset of points with respect to the full tensorization set. This strategy is a cure against the so-called curse of dimensionality [11], i.e. the exponential growth of the number of points with respect to the stochastic dimensions. Relevant works in this context are related to the application of the sparse grid technique for cubature in high-dimensional spaces [23,24], and application to UQ problems [19,35,31]. Note that all these techniques are *non-intrusive*, since the statistics of the quantity of interest are obtained by post-processing multiple independent runs of the deterministic code. However, a possible output for partial differential equations could be the entire space–time space. In this case, these techniques can be still used but each spatial or time location is considered independent without the possibility to re-employ the information already acquired.

The Polynomial Chaos (PC) technique has also acquired great popularity in last years. In the original work of Wiener [44], the solution is expanded in a polynomial Hermite basis, the so-called homogeneous chaos expansion. In recent years, Xiu and Karniadakis [48] demonstrated that the optimal convergence can be achieved if orthogonal basis is chosen following the so-called Wiener–Askey scheme. This approach leads to the well-known generalized Polynomial Chaos (gPC) approach. Long-time integration problems could be encountered [43] due to the modification in time of the statistic properties of the solution, which can yield an efficiency loss of the polynomial basis in time. For curing this issue, Gerritsma [22] proposed a time-dependent generalized Polynomial Chaos scheme. Handling custom-defined probability density function (for example discontinuous and unsteady) is also a limitation of this technique (see for instance [45,37]). Other approaches based on Galerkin projection scheme are the ones which employ a sparsification of the tensor product between the stochastic and spatial space. For instance, in [13] a sparse deterministic–stochastic tensor Galerkin finite element method has been proposed, while in [12] its collocation counterparts have been formulated. Both the previously mentioned approaches are based on an *a priori* selection of the deterministic and stochastic discretization spaces for elliptic pde. As a consequence, even if these ones constitute some attempts for balancing the deterministic and stochastic discretization errors, they are limited to steady and smooth problems.

The major drawbacks encountered in real-like applications for PC, and in general approaches based on the Galerkin projection, are the presence of discontinuities, in both physical and stochastic spaces. Wan and Karniadakis introduced an adaptive class of methods for solving discontinuities by using local basis functions, namely the multi-element generalized Polynomial Chaos (ME-gPC) [18]. This strategy deals with an adaptive decomposition of the domain on which local basis is employed. In order to treat discontinuous response surfaces, Le Maître et al. applied a multiresolution analysis to Galerkin projection schemes [33,32,42]. This class of schemes relies on the projection of the uncertain data on a multi-wavelets basis consisting of piecewise polynomial (smooth) functions. This approach is shown to be very CPU demanding. As a consequence, two cures are then explored in the context of adaptive methods: automatically refine the multi-wavelets basis or adaptively partitioning the domain.

In the context of sparse grid techniques, the issue of discontinuities has also been treated. For curing this issue, in [30] efficient algorithms to track the discontinuities in multidimensional spaces have been proposed. The idea is to apply standard sparse grid based techniques on each sub-domain. The algorithm is based on the polynomial annihilation techniques [39] and becomes optimal for discontinuities residing on a manifold of smaller dimensionality. Even in this case, the method is proposed for the functional approximation problem, while its application to the statistical characterization of random fields is not developed.