Engineering Analysis with Boundary Elements 37 (2013) 1052–1058

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Conservative multiquadric quasi-interpolation method for Hamiltonian wave equations ☆

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ARTICLE INFO

Article history:
Received 6 February 2013
Accepted 23 April 2013
Available online 21 May 2013

Keywords:
Meshless method
Quasi-interpolation
Energy conservation
Hamiltonian wave equations
Symplectic integrator

ABSTRACT

Hamiltonian PDEs have some invariant quantities such as energy and momentum, etc., which should be well conserved with the numerical integration. In this paper we concentrate on the nonlinear wave equation. We study how a space discretization by using multiquadric quasi-interpolation method makes the space discretized system also possess some invariants which are good approximation of the continuous energy. Then, appropriate symplectic scheme is employed for the integration of the semi-discretized system. Theoretical results show that the proposed method has not only high order accuracy but also good properties of long-time tracking capability. Some numerical examples are presented to demonstrate the effectiveness of the proposed method.

1. Introduction

Hamiltonian wave equations can be treated as Hamiltonian systems (in infinite dimensions) [20]. Hamiltonian formalism has the important property of being area-preserving (symplectic). To solve these Hamiltonian PDEs numerically, one hopes that the numerical solution will hold this property too. A standard method to obtain symplectic scheme for an infinite-dimensional Hamiltonian PDEs is that, first discretize the Hamiltonian PDEs in space to obtain a finite-dimensional Hamiltonian system, and then evolve the semi-discrete system by symplectic integrators [2]. In this numerical procedure, the key for success is to ensure that the obtained semi-discrete system is a finite-dimensional Hamiltonian ODEs system, for which finite difference method (FDM) [6,11], finite element method (FEM) [27], Fourier pseudospectral method [13] can be utilized. However, most of those methods depend on a suitable generation of meshes, which is difficult for problems with very complicated and irregular geometries. To develop a meshless symplectic integrator or meshless energy-conserving numerical scheme on scattered nodes motivates the current work.

It is well known that the multiquadric is one of the most often applied kernels in meshless methods. Multiquadric kernels were proposed by Hardy [9]. Franke designed lots of numerical experiments, among which multiquadrics performed best [8]. Therefore multiquadric quasi-interpolation method has caught the attentions of many researchers. For the meshless collocation (or interpolation) method for PDEs by using multiquadric functions, one is required to solve a large scaled linear system of equations; moreover, the coefficients matrix is usually very ill-conditioned and the results are sensitive to the shape parameter c [16]. The most important advantage of quasi-interpolation is that one can evaluate the approximant directly without needs to solve any linear system of equations. Beatson and Powell first proposed some quasi-interpolation scheme by using multiquadric [1]. Beatson even used the multiquadric quasi-interpolation as a computer aided design tool in the film "The Lord of the Rings III". Ref. [23] improved these schemes and discussed their approximation order and the shape preserving property. Lately [12] proved that multiquadric quasi-interpolation can approximate not only the function itself but also its high order derivatives. Ref. [24] used the multiquadric quasi-interpolation to solve free boundary diffusion problem. The multiquadric kernel method is one of the radial basis functions (RBFs) methods. RBFs method for solving PDEs has become one part of the new numerical meshless methods. More details about RBFs meshfree approximation methods for PDEs can be found in [5].

To be more precise, define the multiquadric function $\phi(x)=\sqrt{x^2+c^2}$ and $\phi_j(x)=\phi(x-x_j)$, where c is a shape parameter. Multiquadric quasi-interpolation of a function $f:\mathcal{R}\mapsto\mathcal{R}$ on the scattered knots

$$\cdots<x_{-1}<x_0<x_1<\cdots<x_N<\cdots,\quad h:=\max_j(x_j-x_{j-1}),$$

takes the form

$$(\mathcal{L}f)=\sum f(x_j)\psi_j(x), \tag{1.1}$$

☆Research partially supported by Grant 12DZ2272800.

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http://dx.doi.org/10.1016/j.enganabound.2013.04.011

where $\psi_j(x)$ are the following linear combinations of the multiquadrics, that

$$\psi_j(x) = \psi(x - x_j) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}.$$

The purpose of this paper is to present a meshless energy-conserving numerical method by using multiquadric quasi-interpolation as an approximation scheme for numerical solution of Hamiltonian wave equation

$$u_{tt} - u_{xx} + F'(u) = 0, \tag{1.2}$$

where $F : \mathcal{R} \mapsto \mathcal{R}$ is a smooth function.

An outline of the paper is as follows. In Section 2, preliminaries about Hamiltonian wave equation and the properties of multiquadric function are recalled. In Section 3 we will discuss how a space discretization by using multiquadric quasi-interpolation method makes the space discretized system that also has some invariants which well approximate the continuous energy. In Section 4, the conservative multiquadrics quasi-interpolation method is introduced by using *staggered Störmer–Verlet scheme*

$$V^{n+1/2} = V^{n-1/2} + \tau \Phi_2 M U^n - \tau F'(U^n)$$

$$U^{n+1} = U^n + \tau V^{n+1/2}.$$

This scheme conserve a quantity $\tilde{H}_\Delta$ satisfying

$$H_\Delta = \tilde{H}_\Delta + \mathcal{O}(\tau^2).$$

Both the truncation error and globe error are also studied. In Section 5, numerical examples are tested to verify the effect of the method. Finally, concluding remarks show that by further study the proposed method can be applied to construct not only conservative moving knots but also high-order schemes.

2. Preliminaries

2.1. Nonlinear wave equation

We consider the nonlinear wave equation (1.2). This equation is used to model nonlinear phenomena such as the propagation of dislocation in crystal and the behavior of elementary particles. It is also used in soliton theory. The equation is a classical example of Hamiltonian PDEs (infinite-dimensional Hamiltonian system) [20]. By defining a new variable $v = u_t$, the Hamiltonian formulation goes as

$$\begin{cases} u_t = +\dfrac{\delta H}{\delta v} = v \\ v_t = -\dfrac{\delta H}{\delta u} = u_{xx} - F'(u), \end{cases} \tag{2.1}$$

where

$$H(u,v) = \frac{1}{2}\int [v^2 + u_x^2 + 2F(u)]\, dx \tag{2.2}$$

is invariant with respect to time under an appropriate initial boundary-value condition. Here $\delta H/\delta u$, $\delta H/\delta v$ are the variational, or Gateaux derivatives defined by

$$\left(\frac{d}{d\epsilon}H[u+\epsilon\phi]\right)_{\epsilon=0} \equiv \int \frac{\delta H}{\delta u}\phi\, dx, \quad \left(\frac{d}{d\epsilon}H[v+\epsilon\phi]\right)_{\epsilon=0} \equiv \int \frac{\delta H}{\delta v}\phi\, dx.$$

The symplectic form of this system

$$\Omega = \int du \wedge dv\, dx \tag{2.3}$$

is also invariant with respect to time. More details can be found in [13,15].

2.2. Discretization method for the NLW equation

Classical methods to solve (2.1) numerically are those, where a standard procedure starts with the discretization of the equation in space and then in time. After the discretization in space, the following semi-discretized problem arises:

$$\begin{cases} \dfrac{d}{dt}U_h = V_h \\ \dfrac{d}{dt}V_h = A_h U_h - F'(U_h) \end{cases} \tag{2.4}$$

where $U_h(t) = (\ldots, u(jh,t), \ldots)^T$. To preserve the symplectic form of (2.1), an appropriate numerical discretization scheme needs to be developed in the sense that the above resulting semi-discrete system (continuous in time) can be written as a finite-dimensional Hamiltonian system. For this purpose, the numerical scheme is required to be able to preserve the symmetric property of second-order differential operator embedded in (2.1). That means a suitable A_h must be symmetric [3]. Several methods can be chosen such as the finite difference method [2], finite element method [27] and Fourier pseudospectral method [13] on a uniform grid.

As for nonuniform knots, consider the finite divided difference approximation of u_{xx} at point x_i

$$(u_{xx})_i = \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} - \frac{u_i - u_{i-1}}{x_i - x_{i-1}}\right) \Big/ \frac{x_{i+1} - x_{i-1}}{2},$$

by defining $\Delta x_i = (x_{i+1} - x_{i-1})/2$, $\alpha_i = 1/(x_i - x_{i-1})$, $\beta_i = 1/x_{i+1} - x_i$, and

$$M := \begin{pmatrix} \ddots & \alpha_i & \\ \alpha_i & -(\alpha_i + \beta_i) & \beta_i \\ & \beta_i & \ddots \end{pmatrix},$$

taking $\tilde{u}_i = \sqrt{\Delta x_i}\, u_i$, we can get

$$\begin{pmatrix} \vdots \\ (\tilde{u}_{xx})_i \\ \vdots \end{pmatrix} \approx \begin{pmatrix} \ddots & & \\ & \frac{1}{\sqrt{\Delta x_i}} & \\ & & \ddots \end{pmatrix} M \begin{pmatrix} \ddots & & \\ & \frac{1}{\sqrt{\Delta x_i}} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \tilde{u}_i \\ \vdots \end{pmatrix}.$$

Sometimes the sampling data points (knots) should be moved according to the equation, e.g. [10,21]; it is hard to solve the nonlinear partial differential propagations equation with moving knots by using the traditional methods which depend on a suitable generation of meshes. Multiquadric quasi-interpolation method is a true meshless method, it can be used for constructing moving knots schemes [21] and can be generalized to high-dimensional space [22]. In this paper, a multiquadric quasi-interpolation method will be used for the spatial discretization of the wave equation.

2.3. The Multiquadric function

Multiquadric function satisfies $\int \phi''(x)/2\, dx = 1$, some important properties are given in the following lemmas. Based on the approach of Cheney [4,12] Lemmas 2.1 and 2.2 were proved.

Lemma 2.1. *If* $f \in \mathcal{C}^2(\mathcal{R})$, *then the following inequality*

$$\left| \int_{-\infty}^{+\infty} f(t)\frac{\phi''(x-y)}{2}\, dt - f(x) \right| \le \mathcal{O}(c^2) \tag{2.5}$$

holds.

Lemma 2.2. *If* $f(x) \in \mathcal{C}^2(\mathcal{R})$ *and* f, f' *and* f'' *are bounded by a polynomial of degree*2, 1 *and* 0, *respectively, then the following inequality*

$$\left| \int \frac{\phi''(x-t)}{2} f(t)\, dt - \sum_j f(x_j)\frac{\phi''(x-x_j)}{2}\Delta_j \right| < \mathcal{O}(h/c) \tag{2.6}$$

holds, where $\Delta_j = (x_{j+1} - x_{j-1})/2$.