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A new finite volume approach for transport models and related applications with balancing source terms

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Abstract

We develop a new finite volume scheme for numerically solving transport models associated with hyperbolic problems and balance laws. The numerical scheme is obtained via a Lagrangian–Eulerian approach that retains the fundamental principle of conservation of the governing equations as it is linked to the classical finite volume framework. As features of the novel algorithm we highlight: the new scheme is locally conservative in balancing the flux and source term gradients and preserves a component-wise structure at a discrete level for systems of equations. The novel approach is applied to several nontrivial examples to evidence that we are calculating the correct qualitatively good solutions with accurate resolution of small perturbations around the stationary solution. We discuss applications of the new method to classical and nonclassical nonlinear hyperbolic conservation and balance laws such as the classical inviscid Burgers equation, two-phase and three-phase flow problems in porous media as well as numerical experiments for nonlinear shallow water equations with friction terms. In addition, we consider the case of the source term which is discontinuous as a function of space x. We also extend the Lagrangian–Eulerian framework to the two-dimensional scalar conservation law, along with pertinent numerical experiments to show the performance of the new method.

Keywords: Conservation laws; Balance laws; Flow in porous media; Shallow water equations; Lagrangian–Eulerian finite volume

1. Introduction

Our aim is to develop a simple and efficient class of finite volume schemes based on a novel Lagrangian–Eulerian framework to account the delicate nonlinear balance between the discretizations of the hyperbolic flux and of the stiff source term at a discrete level. A rigorous mathematical demonstration of such approach is beyond the scope of the present paper, and is to be attempted in future research. In the current stage of this work, we describe the key ideas of our constructive algorithm and we present several nontrivial numerical experiments in order to verify the desired

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well-balanced properties for one-dimensional and two-dimensional problems involving conservation laws with source terms for transport models and related applications.

It is well known that many well-balanced schemes have been proposed since the milestone work (see [21]) of Greenberg and Leroux (1996). The focus of many such works was to handle shallow water equations over non-trivial topographies. The key issue is the construction of well-balanced nonlinear schemes that recover the time-asymptotic behavior of the underlying nonlinear balance law. There are many relevant studies for approximation methods and numerical analysis devoted to balance law and hyperbolic law problems. Naturally, all methods exhibit advantages and disadvantages, since the underlying differential equations are very hard problems with a lack of general theory (see, e.g., [6–9,12,13,15,19,22,21,25,28,30,33,35]). See also [4,17] for surveys on both analytical and numerical aspects of one-dimensional hyperbolic balance laws and [18] for a good discussion of two-dimensional balance law problems along with an up-to-date and comprehensive list of references. The work [18] also includes relevant theoretical aspects of scalar conservation laws in several spatial dimensions in a more flexible Godunov framework to handle local nonlinear wave patterns to account for the flux computations. These schemes evolved following the natural understanding of fundamental concepts from the theory of nonlinear hyperbolic conservation laws concerning the properties of the characteristic surfaces, such as existence, uniqueness, and solution of the Riemann problems. Also, for a scalar balance law, the solution depends strongly on certain properties of the source term (see, e.g., [5,17, 19,21,24,25]).

The situation of balance laws $u_t + f_x(u) = g(x, u)$, with $g(x, u)$ discontinuous in x is another challenging problem encountered for such class of differential equations from both a theoretical and a numerical point of view. A kinetic scheme, with convergence proof for the scalar related problem, was introduced in [7]. Essentially, they propose a kinetic interpretation of upwinding techniques, taking into account the source terms to develop an equilibrium scheme as a result. Another approach was introduced in [35] (see also [8]). This method does not use upwinding solvers; it uses the interface value rather than the cell averages for the source terms that balance the nonlinear convection at the cell interface, allowing the numerical capturing of the steady state with a formal high order accuracy. A successful alternative to accounting for the balance between the nonlinear flux and the source terms with $g(x, u)$ discontinuous in x is the use of a central differencing scheme as discussed in [2] for gas dynamics Euler equations with stiff relaxation source terms; see also [8]. A distinct numerical framework, based on Riemann solvers using local characteristic decompositions, can be found in [15]. In paper [16], the authors were concerned with the Riemann problem of the Burgers equation with a discontinuous source term, motivated by the study of propagation of singular waves in radiation hydrodynamics. Moreover, they were able to construct the global entropy solution to the related Riemann problem linked to this model. It turns out that the discontinuity of the source term has clear influences on the shock or rarefaction waves generated by the initial Riemann data. It is worth mentioning that other related problems were also described in the literature supported by numerical experiments. For more details about this subject matter, the interested reader is referred to the papers [2,4,7,8,15–17,35] and papers cited therein.

In this work we are interested in the construction of a numerical scheme for solving nonlinear hyperbolic conservation and balance law problems using a Lagrangian–Eulerian approach [5,14,24,25]. In the work [14], the authors identified the region in the space–time domain where the mass conservation takes place, but linked to a scalar convection-dominated nonlinear parabolic problem, which models the immiscible incompressible two-phase flow in a porous medium. The key ingredient to finding this conservative region was the use of a Lagrangian–Eulerian framework; see [5] for related works with applications to radionuclide transport problems. Recently in [33], such ideas were extended to nonlinear purely hyperbolic conservation and balance laws—scalar case and systems of equations. In particular, a convergence proof for the unique entropy solution was established for the case of a Lagrangian–Eulerian monotone finite difference scheme related to a scalar hyperbolic conservation law. We refer to [3] for other recent developments on this subject.

We will explore the above mentioned innovative ideas to give a formal construction of accurate Lagrangian–Eulerian schemes for transport models and related applications with balancing source terms. As features of the novel algorithm, we highlight: we verified through numerical experiments that the new scheme seems to be locally conservative in balancing the flux and source term gradients and preserves a component-wise structure at the discrete level for systems of equations. Besides, we also discuss a set of numerical experiments to nonlinear scalar two-dimensional problems with non-symmetric and nonconvex flux function for systems of balance laws. This novel approach is applied to several nontrivial examples to show evidence that we are calculating the correct qualitatively good solutions with the accurate resolution of small perturbations around the stationary solution. We