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Uniformly stable wavelets on nonuniform triangulations

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Abstract

In this paper we construct linear, uniformly stable, wavelet-like functions on arbitrary triangulations. As opposed to standard wavelets, only local orthogonality is required for the wavelet-like functions. Nested triangulations are obtained through refinement by two standard strategies, in which no regularity is required. One strategy inserts a new node at an arbitrary position inside a triangle and then splits the triangle into three smaller triangles. The other strategy splits two neighbouring triangles into four smaller triangles by inserting a new node somewhere on the edge between the triangles. In other words, non-uniform refinement is allowed in both strategies. The refinement results in nested spaces of piecewise linear functions. The detail-, or wavelet-spaces, are made to satisfy certain orthogonality conditions which locally correspond to vanishing linear moments. It turns out that this construction is uniformly stable in the L_∞ norm, independently of the geometry of the original triangulation and the refinements.

Keywords: Wavelets; Approximation; Stability; Non-uniform triangulations

1. Introduction

Wavelets have become a popular tool in many areas of mathematics and science. Classical wavelets were defined on regular uniform grids over the whole real line and were required to satisfy strong orthogonality conditions [4]. Early extensions relaxed the orthogonality conditions and provided simple restrictions to intervals, cf, [2]. The use of spline wavelets provided better treatment of boundary conditions on intervals, as well as a natural construction of wavelets on non-uniform grids, as shown in [1,3,9].

Any univariate construction, including wavelets, can be extended to the multivariate setting by the standard tensor product construction. Various kinds of wavelets have also been constructed on triangulations, but to our knowledge, the most general setting for these constructions is a non-uniform base triangulation with some kind of uniform refinement rule, see e.g. [5–7,10,11].

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Construction of wavelets over irregular grids raises an additional issue, namely whether the construction is stable independently of the grid geometry. It was recently shown in [8] that this is indeed the case for univariate, linear wavelets on irregular grids with vanishing moments when the stability is measured in the uniform norm.

The purpose of the present paper is to generalise the results in [8] to linear wavelets over general triangulations. Linear wavelets that are locally orthogonal to the original basis of hat functions are constructed. We use two standard, but not widely used, refinement rules, which both allow non-uniform refinement. These wavelets are shown to be uniformly stable, independently of the topology and geometry of both the original triangulation and the refinements. As in [8] we measure stability in the uniform norm. We limit our studies to general triangulations that can be projected onto a plane.

In Section 2 we give a brief overview of the construction. In Section 3 we discuss the first refinement strategy in detail, including stability results, and in Section 4 we discuss the second strategy. In Section 5 we then combine these results and consider iterated refinement with a combination of the two strategies. We end with some examples in Section 6 and conclude in Section 7.

2. An overview of the wavelet construction

Let N be a finite set of points in $\mathbb{R}^2$, usually referred to as nodes. Any set of three nodes forms a triangle, and a triangulation Δ over N is a collection of triangles with the property that two triangles in Δ are either disjunct, or have a vertex or edge in common. We will refer to the number of edges emanating from a node as its valence. For each node $\boldsymbol{v} \in N$ we construct the linear B-spline (hat function) $\phi_{\boldsymbol{v}}$ with the property that for any two nodes $\alpha, \beta \in N$ we have $\phi_\alpha(\beta) = \delta_{\alpha\beta}$.

We start with an arbitrary base triangulation Δ_0 defined over an initial set N_0 of nodes. We then refine the base triangulation through node insertions, where each node is inserted according to one of two alternative strategies. The first strategy is to insert a new node $\boldsymbol{p}$ in the interior of a triangle $T = (\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2)$ and split the triangle into three smaller triangles, as shown in Fig. 1(a). The inserted point $\boldsymbol{p}$ can then be expressed as a convex combination of $\boldsymbol{v}_0, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ by $\boldsymbol{p} = a_0\boldsymbol{v}_0 + a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2$, where $\boldsymbol{a} = (a_0, a_1, a_2)$ contains the barycentric coordinates of the point $\boldsymbol{p}$, i.e., they satisfy $a_i \geq 0$ and $\sum_{i=0}^{2} a_i = 1$. For $\boldsymbol{p}$ to be inserted inside the triangle, we require $0 < a_i < 1$. The second strategy for node insertion is to insert the new node $\boldsymbol{p}$ along an edge $e = (\boldsymbol{v}_0, \boldsymbol{v}_1)$ and divide each of the two triangles sharing the edge into two new triangles, as shown in Fig. 1(b). The new node can now be expressed as $\boldsymbol{p} = \lambda\boldsymbol{v}_0 + (1-\lambda)\boldsymbol{v}_1$, where $0 < \lambda < 1$. Regardless of the insertion strategy, we can construct a new hat function $\sigma_{\boldsymbol{p}}$, such that $\sigma_{\boldsymbol{p}}(\boldsymbol{p}) = 1$ and $\sigma_{\boldsymbol{p}}(\boldsymbol{v}) = 0$ for all nodes $\boldsymbol{v} \in N_0$. In either case we denote the new set of nodes $N_0 \cup \{\boldsymbol{p}\}$ by N_1 and the new triangulation by Δ_1.

If we allow one or more $a_i \in \{0, 1\}$ or $\lambda \in \{0, 1\}$ for an inserted knot $\boldsymbol{p}$, the corresponding hat function $\sigma_{\boldsymbol{p}}$ will be discontinuous. For simplicity we will not discuss these cases in this paper.

We will now give an overview of our wavelet construction for node insertion strategy 1. Strategy 2 is treated later in a similar way. The set $\boldsymbol{\phi} = \{\phi_{\boldsymbol{v}} \mid \boldsymbol{v} \in N_0\}$ forms a basis for the space $\mathbb{V}_0 = \mathbb{V}(\Delta_0)$ of continuous functions that are linear on each triangle in Δ_0. Similarly, the set $\boldsymbol{\gamma} = \{\gamma_v \mid v \in N_1\}$ forms a basis for the refined space $\mathbb{V}_1$, and it is well-known that $\mathbb{V}_0 \subseteq \mathbb{V}_1$. The natural generalisation of the construction in [8] is to construct an alternative basis $\{\boldsymbol{\phi}, \hat{\psi}_{\boldsymbol{p}}\}$ for $\mathbb{V}_1$ with the property that

$$\int_{\mathbb{R}^2} \hat{\psi}_{\boldsymbol{p}}\, g = 0, \quad \text{for } g = 1, x, y.$$

Here $\hat{\psi}_{\boldsymbol{p}} = \gamma_{\boldsymbol{p}} - \sum_{i=0}^{2} c_i \phi_{\boldsymbol{v}_i}$, where $\boldsymbol{v}_i$ are the vertices of the triangle that contains $\boldsymbol{p}$, and $(c_i)_{i=0}^{2}$ are certain coefficients $(c_i)_{i=0}^{2}$ to be determined. These equations constitute a linear system of equations for determining the unknown coefficients, but unfortunately, it turns out that this construction is not stable independently of the geometry. More specifically, there exist triangulations such that the resulting linear system of equations is singular. An example of such a triangulation is shown in Fig. 2.

We want to construct an alternative basis $\{\boldsymbol{\phi}, \psi_{\boldsymbol{p}}\}$ for $\mathbb{V}_1$ with the property that the function $\psi_{\boldsymbol{p}}$ satisfies the orthogonality condition

$$\int \phi_{\boldsymbol{v}} \psi_{\boldsymbol{p}} = 0 \tag{1}$$