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Simulations of non homogeneous viscous flows with incompressibility constraints

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Abstract

This presentation is an overview on the development of numerical methods for the simulation of non homogeneous flows with incompressibility constraints. We are particularly interested in systems of partial differential equations describing certain mixture flows, like the Kazhikhov–Smagulov system which can be used to model powder-snow avalanches. It turns out that the Incompressible Navier–Stokes system with variable density is a relevant step towards the treatment of such models, and it allows us to bring out some interesting numerical difficulties. We should handle equations of different types, roughly speaking transport and diffusion equations. We present two strategies based on time-splitting. The former relies on a hybrid approach, coupling finite volume and finite element methods. The latter extends discrete duality finite volume schemes for such non homogeneous flows. The methods are confronted to exact solutions and to the simulation of Rayleigh–Taylor instabilities.

Keywords: Non homogeneous viscous flows; Navier–Stokes equations; Mixtures; Multifluid flows; Finite volume methods

1. Introduction

The motivation of this work comes from the mathematical modeling of mixtures or multifluid flows. In fact, there exists a large variety of partial differential equations (PDEs) systems intended to describe such flows, and, of course, we will focus on some specific models. The situations we are interested in can be characterized by the following features:

- There are strong density variations in the flows, and the numerical challenge is to capture and to follow with accuracy the strong gradients and the fronts of density variations.
- The set of equations involves a constraint on the divergence of the velocity field, hereafter denoted by $\mathbf{u}$. The simplest of these constraints is the solenoidal condition $\mathrm{div}_x(\mathbf{u}) = 0$, but we shall see more intricate situations.

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Our objective consists in designing dedicated numerical methods for the simulations of these models. The difficulty comes from the fact that the system couples equations of different type, and the constraint mentioned above. The constraint relies on modeling assumptions. In order to set up performing methods, it could be helpful to understand the origin of this relation. We will give some hints in this direction. The discussion should be taken with full awareness that many aspects in the derivation of the equations are not that neat, and can be considered as questionable: mathematical models in this field remain under debate. According to prescriptions in [60], our viewpoint is therefore fully pragmatic: let us say that we are just picking a set of equations, and we try to discuss some mathematical properties, and to set up specific numerical schemes that allow us to investigate the sensitivity of the model with respect to variations of the (physical and numerical) parameters. However, we should bear in mind that the robustness of the conclusion should be considered with caution: changing a "detail" in the modeling assumptions might dramatically affect the mathematical structure of the model, which thus would require another approach.

Let us end this introduction with a few words about potential applications of our approaches. Powder-snow avalanches is a relevant example of the kind of fluid mixtures we are considering and variations about the Navier–Stokes system have been used successfully to reproduce certain features of laboratory avalanches [26–28]. These complex models also naturally arise in combustion theory; nuclear safety engineering provides further relevant examples of applications, see for instance the works [24,6,59] motivated by security computations for PWR reactors. Finally, we mention that the reasonings of mixture theory have been developed to derive models describing biofilms formation [17].

The paper is organized as follows. We start by discussing a few relevant mathematical models that couple transport, diffusion and constraint equations. In Section 3 we present the hybrid finite volume (FV)–finite element (FE) scheme developed in [14,12,15]. The method is designed to reach the second order accuracy, at least for smooth solutions, owing to a suitable adaptation of Monotonic Upstream-Centered Scheme for Conservation Laws (MUSCL) techniques for the transport equation. Here, we pay attention to the treatment of the boundary conditions. Proceeding naively might lead to spurious instabilities when the geometry of the mesh and of the computational domain is non trivial. An alternative numerical method based on the Discrete Duality Finite Volume (DDFV) framework is introduced in Section 4. The DDFV framework is an attempt to cope with the difficulty in defining diffusion fluxes on interfaces without imposing geometric restrictions on the mesh construction, see [42,43,25]. The method has been extended to deal with Stokes and homogeneous Navier–Stokes equations [22,47,49,48]. Dealing with non homogeneous flows requires a specific attention to design a relevant treatment of the convection terms of the system [33]. Section 5 offers a set of numerical experiments where we compare the performances of the two methods. In particular, we bring out difficulties related to the treatment of the boundary terms and the sensitivity to the mesh construction.

2. Examples of non-homogeneous flows involving constraints on $\mathrm{div}_x(\mathbf{u})$

2.1. *Incompressible flows*

Let us start by recalling a few facts about the simplest situation of incompressible flow which means that the velocity is required to satisfy

$$\mathrm{div}_x(\mathbf{u}) = 0. \tag{1}$$

Neglecting any difficulty associated to a possible lack of regularity of $\mathbf{u} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, let us consider the characteristic curves, defined by the ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, x) = \mathbf{u}(t, X(t, x)), \qquad X(0, x) = x.$$

The quantity $X(t, x) \in \mathbb{R}^N$ is nothing but the position at time $t \geq 0$ of a particle, driven by the velocity field $\mathbf{u}$, which starts at time $t = 0$ from the position $x \in \mathbb{R}^N$. Let us consider a fixed domain D_0, and consider its image at a latter time $t > 0$

$$D(t) = \{X(t, x_0),\ x_0 \in D_0\}.$$