Engineering Analysis with Boundary Elements 37 (2013) 1117–1126

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

An implicit meshless scheme for the solution of transient non-linear Poisson-type equations

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ARTICLE INFO

Article history:
Received 2 November 2012
Accepted 4 April 2013
Available online 23 May 2013

Keywords:
Meshfree point collocation method
MLS
Nonlinear Poisson equation
Linearization method
Lagging coefficients

ABSTRACT

A meshfree point collocation method is used for the numerical simulation of both transient and steady state non-linear Poisson-type partial differential equations. Particular emphasis is placed on the application of the linearization method with special attention to the lagging of coefficients method and the Newton linearization method. The localized form of the Moving Least Squares (MLS) approximation is employed for the construction of the shape functions, in conjunction with the general framework of the point collocation method. Computations are performed for regular nodal distributions, stressing the positivity conditions that make the resulting system stable and convergent. The accuracy and the stability of the proposed scheme are demonstrated through representative and well-established benchmark problems.

1. Introduction

Non-linearity is inherent in the majority of physical phenomena and engineering processes, resulting in non-linear ordinary or partial differential equations. In fact, these non-linear equations are usually difficult to solve, since no general technique works globally and, thus, each individual equation has to be studied as a separate problem. Of particular interest are the non-linear Poisson-type equations that are encountered in transport problems, notably heat conduction, mass transport and flow in different geometries, including pore structures.

Solutions to this type of problems are usually required in non-regular two- and three-dimensional geometries with non-uniform boundary conditions. The non-linear character of the partial differential equations that govern these problems renders the analytical solution either difficult or impossible, especially when dealing with irregular geometries and complex boundary conditions. The use of conventional numerical procedures, such as finite differences [1], finite elements [2], boundary element method [3], etc., to solve these problems necessitates high levels of discretization, resulting in large computational time. In fact, the difficulty of the finite difference method to deal with problems with irregular geometry is a major drawback of the method. On the other hand, the finite element methods do handle irregular geometries, yet the refinement procedure becomes a major task.

To overcome these drawbacks, new sophisticated numerical methods have appeared over the last years [4,5]. Meshless (or meshfree) methods emerged as potential candidates to replace traditional numerical methods for problems where the latter failed to provide accurate or fast results. Thus, the challenges that these methods have to face are both the efficiency and the accuracy of computations. In a general sense, a numerical method must be robust and accurate in order to be able to provide numerical results in real time when dealing with real world applications. Additionally, they have to deal with the complexity of the phenomena under consideration, which makes the computations even more challenging. In fact, in cases where no analytical solutions are available, the refinement procedure becomes a necessity and, therefore, an easy and efficient refinement procedure must be available. Despite the tremendous improvement of the meshless methods, they are still at their early stages of development and, thus, several test studies must be conducted in order for these methods to establish themselves as more general practical tools ([6] and references there in). Among the existing meshless approximation/interpolation methods, both Moving Least Squares (MLS) and Radial Basis Functions (RBF) have been widely used to solve numerous challenging problems [7]. Meshfree methods based on MLS have a clear advantage over other meshfree methods thanks to the simplicity and stability of the MLS method in field variable approximation/interpolation [4,5]. The meshless methods provide accurate numerical results

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http://dx.doi.org/10.1016/j.enganabound.2013.04.003

when dealing with certain Poisson-type problems, linear and non-linear, in two and three dimensions [8–13]. The governing equations can be solved in their weak form [10–12] or in their strong formulation [8,9,13]. Despite their accuracy, the former methods deal with the computation of integrals and may prove very time demanding and inaccurate, whereas the latter lead to dense matrices that can be difficult to solve when dealing with real world applications with increased number of degrees of freedom. Despite their computational cost, thanks to their key advantage of being applicable to arbitrarily irregular geometries, they are attractive for solving non-linear Poisson type problems.

In this work, a meshfree point collocation method is used for the numerical simulation of both transient and steady state non-linear Poisson-type partial differential equations with the objective to extend the applicability of the method to a wider range of non-linear differential equations. The strong form equations are solved with the collocation method. The implementation is easy and straightforward and, the resulting linear system is sparse and positive definite. Thus, the numerical solution of the harmonic operator can be accurate and fast, taking advantage of the sparsity of the matrix. The article is organized as follows. In Section 2, the governing equations are presented along with a complete description of the MLS approximation scheme. Section 3 presents the meshless point collocation transient, coupled solver along with a θ-weighted time discretization approach, which is employed to solve numerically the partial differential equations. Numerical issues are discussed in Section 4, where typical transient and steady state physical problems are solved and the validity of the proposed meshless techniques is demonstrated. Finally, in Section 5, the main conclusions are discussed.

2. Mathematical problem and approximation procedure

2.1. Governing equations

Consider the following partial differential equation:

$$\frac{\partial u}{\partial t} = \Delta u + f(\boldsymbol{x}, u, u_{,x}, u_{,y}, u_{,xx}, u_{,xy}, u_{,yy}) \tag{1}$$

for the physical field u, subject to the following boundary and initial conditions:

- Dirichlet boundary condition for the unknown field:
  $u = u_D$ on $\partial\Omega^D$
- Neumann boundary condition for the component of the field-gradient normal to the boundary:
  $q = q_N$ on $\partial\Omega^N$
- Mixed (Robin) boundary condition for the component of the field-gradient normal to the boundary:
  $au + \beta q = B_R$ on $\partial\Omega^R$
- Initial condition at $t=0$:
  $u = u_0$ on Ω

where $\boldsymbol{x}\in\mathbb{R}^d$ is the position vector, $d=2$ is the dimension of domain $\Omega(x,y)$, which has a piecewise smooth boundary $\partial\Omega$. Δ represents the Laplacian operator, and q stands for the boundary normal flux defined by $q=-\partial u/\partial \boldsymbol{n}$, with $\boldsymbol{n}$ the unit outward normal to the boundary $\partial\Omega(\partial\Omega=\partial\Omega^D\cup\partial\Omega^N\cup\partial\Omega^R)$, u_D, q_N are specified values on the boundary, respectively, whereas α, β, B_R are known coefficients. Eq. (1) applies to a wide range of engineering problems that distinguish themselves through the type of the right-hand side function f. More precisely, Eq. (1) can be reduced to a standard Poisson equation or Helmholtz equation if f is a function of position vector $\boldsymbol{x}$ or a linear function of physical field u, respectively. More generally, f may be a nonlinear function of physical field u and its derivatives. The method that is presented below is able to handle these types of problems.

2.2. Moving Least Squares shape functions

In the context of the meshless approximation/interpolation schemes, the MLS method [14] is widely used, since it can directly approximate the field variables in a local manner and, additionally, can be easily extended to n-dimensional problems. A brief summary of the MLS approximation scheme is given next.

Within the MLS context, the approximation $u^h(\boldsymbol{x})$ of the unknown field function $u(\boldsymbol{x})$ is expressed as:

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{m} p_i(\boldsymbol{x})\alpha_i(\boldsymbol{x}) = \boldsymbol{p^T}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) \tag{2}$$

where $\boldsymbol{p^T}(\boldsymbol{x})$ is a polynomial basis in the space coordinates, that consists most often of monomials of the lowest order to ensure completeness, m is the total number of the terms in the basis and, $\boldsymbol{\alpha}(\boldsymbol{x})$ is the vector of coefficients, given by

$$\boldsymbol{\alpha}(\boldsymbol{x}) = (\alpha_0(\boldsymbol{x}), \alpha_1(\boldsymbol{x}), ..., \alpha_m(\boldsymbol{x}))^T \tag{3}$$

which is a function of $\boldsymbol{x}$. Due to the local nature of the approximation, the polynomial basis can be written as

$$\boldsymbol{p^T}(\boldsymbol{x}-\boldsymbol{x}_i) = [1, (x-x_i), (y-y_i), ..., (x-x_i)(y-y_i)^{m-1}, (y-y_i)^m] \tag{4}$$

in 2D problems. Herein, a second order ($m=6$) polynomial basis is used,

$$\boldsymbol{p^T}(x) = [1, (x-x_i), (y-y_i), (x-x_i)^2, (x-x_i)(y-y_i), (y-y_i)^2] \tag{5}$$

The derivatives of the unknown field function are up to second order. Thus, in the context of strong form collocation the monomial basis has to be at least second order. For our computations we used the lowest (second) order of monomials that ensured both convergence and accuracy for the harmonic operator [5]. There exists a unique local approximation associated with each point in the domain. To obtain the local approximation of the function $u(\boldsymbol{x})$ and determine the form of $\boldsymbol{\alpha}(\boldsymbol{x})$, the difference between the local approximation $u^h(\boldsymbol{x})$ and the function $u(\boldsymbol{x})$ must be minimized. Thus, a weighted discrete error norm,

$$J(\boldsymbol{x}) = \sum_{i=1}^{n} \boldsymbol{w}_i(\boldsymbol{x},\boldsymbol{x}_i)[u^h(\boldsymbol{x})-u(\boldsymbol{x})]^2 = \sum_{i=1}^{n} \boldsymbol{w}_i(\boldsymbol{x},\boldsymbol{x}_i)[\mathbf{p}^T(\boldsymbol{x}_i)\boldsymbol{a}(\boldsymbol{x})-u(\boldsymbol{x}_i)]^2 \tag{6}$$

is constructed and minimized with respect to the vector $\boldsymbol{\alpha}(\boldsymbol{x})$ of coefficients. The weight function $w_i(\boldsymbol{x},\boldsymbol{x}_i)\equiv w(\boldsymbol{x}-\boldsymbol{x}_i)$ is usually built in such a way that it takes a unit value in the vicinity of node i, where the function and its derivatives are to be computed, and vanishes outside a region Ω_i surrounding the point $\boldsymbol{x}_i$ (called the support domain of node i), with n being the number of nodes in the domain. The choice of the weight function $w(\boldsymbol{x}-\boldsymbol{x}_i)$ affects the resulting approximation $u^h(\boldsymbol{x}_i)$ significantly. In the present work, a Gaussian weight function is used [5], yet the support domain does not have a standard point density value. Instead, a constant number of nodes are used for the approximation of the field function. That is,

$$w_i(x) = \begin{Bmatrix} e^{-(D(x)/d_i)^2} & 0 \le \frac{D(x)}{d_i} \le 1 \\ 0 & \frac{D(x)}{d_i} > 1 \end{Bmatrix}, \tag{7}$$