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Recent constructions of low-discrepancy sequences

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Abstract

We present a survey of the recently developed theory of $(u, \mathbf{e}, s)$-sequences which has led to new constructions of low-discrepancy sequences. We also review recent constructions of low-discrepancy sequences by means of ergodic theory.

Keywords: Quasi-Monte Carlo method; Low-discrepancy sequence; Digital sequence; Global function field; Ergodic theory

1. Introduction

Low-discrepancy sequences are basic ingredients of quasi-Monte Carlo methods for numerical integration. The standard error bounds for quasi-Monte Carlo integration, such as the Koksma–Hlawka inequality (see [9, Section 2.5]), guarantee a rate of convergence faster than the Monte Carlo rate of convergence if the integration nodes are taken from a low-discrepancy sequence. The widespread use of quasi-Monte Carlo methods in areas like scientific computing, computational finance, and computer graphics implies a heavy demand for concrete sequences with as low a discrepancy as possible. The explicit construction of low-discrepancy sequences has a long history, with the first construction of a low-discrepancy sequence for any dimension dating back to 1960 (see Halton [6] and Section 3). Many exciting developments in this area have taken place since then, and in the present paper we survey construction principles that were found in the last few years. For a general background on low-discrepancy sequences and quasi-Monte Carlo methods, we refer to the monographs [2] and [11].

Let $s \geq 1$ be a given dimension and let $\mathcal{P}$ be a point set consisting of N points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ in the s-dimensional unit cube $[0, 1]^s$. For an arbitrary subinterval J of $[0, 1]^s$, we introduce the *local discrepancy*

$$R(J; \mathcal{P}) = \frac{A(J; \mathcal{P})}{N} - \lambda_s(J),$$

where $A(J; \mathcal{P})$ is the number of integers n with $1 \leq n \leq N$ such that $\mathbf{x}_n \in J$ and λ_s denotes the s-dimensional Lebesgue measure. Then we define the *star discrepancy*

$$D_N^*(\mathcal{P}) = \sup_J |R(J; \mathcal{P})|,$$

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where the supremum is extended over all subintervals J of $[0, 1]^s$ with one vertex at the origin. For an infinite sequence S of points in $[0, 1]^s$ and any integer $N \geq 1$, let $D_N^*(S)$ be the star discrepancy of the point set consisting of the first N terms of S. We say that S is a *low-discrepancy sequence* if

$$D_N^*(S) = O\left(N^{-1}(\log N)^s\right) \quad \text{for all } N \geq 2,$$

where the implied constant is independent of N. The asymptotic order of magnitude $N^{-1}(\log N)^s$ is the best that can currently be achieved for the star discrepancy of an infinite sequence of points in $[0, 1]^s$. We refer to [2, Section 3.2] for a survey of known lower bounds on the star discrepancy.

The emphasis in this paper is on types of low-discrepancy sequences that are called $(u, \mathbf{e}, s)$-sequences. The theory of $(u, \mathbf{e}, s)$-sequences was developed very recently, and we present the basics of this theory in Section 2. The principal constructions of $(u, \mathbf{e}, s)$-sequences, which use global function fields, are described in Section 3. Some other recent constructions of low-discrepancy sequences are reviewed in Section 4.

2. $(u, \mathbf{e}, s)$-sequences

Many of the classical constructions of low-discrepancy sequences, such as the constructions of Sobol' sequences [29], Faure sequences [3], Niederreiter sequences [17], generalized Niederreiter sequences [31], and Niederreiter–Xing sequences [23,33], are based on the theory of (t, m, s)-nets and (t, s)-sequences introduced in [16]. Expositions of this theory can be found in the monographs [2] and [18] as well as in the recent handbook article [20]. Therefore we will just recall the basic definitions of this theory here, and then we move on to a recent extension of this theory due to Tezuka [32].

Let $b \geq 2$ be an integer which will be called the base in the sequel. For a given dimension $s \geq 1$, a subinterval J of $[0, 1]^s$ of the form

$$J = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}) \tag{1}$$

with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$ is called an *elementary interval in base b*. For integers $0 \leq t \leq m$, a point set $\mathcal{P}$ consisting of b^m points in $[0, 1)^s$ is called a *(t, m, s)-net in base b* if its local discrepancy satisfies $R(J; \mathcal{P}) = 0$ for all elementary intervals $J \subseteq [0, 1]^s$ in base b with $\lambda_s(J) \geq b^{t-m}$. It is obvious that smaller values of the parameter t signify stronger equidistribution properties of a (t, m, s)-net in base b. A generalization of this concept was recently introduced by Tezuka [32], and we define it in the slightly narrower form presented in Hofer and Niederreiter [7]. Here and in the following, $\mathbb{N}$ denotes the set of positive integers.

Definition 1. Let $b \geq 2$, $s \geq 1$, and $0 \leq u \leq m$ be integers and let $\mathbf{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$. A point set $\mathcal{P}$ of b^m points in $[0, 1)^s$ is called a *$(u, m, \mathbf{e}, s)$-net in base b* if for every elementary interval J in base b of the form (1) with $e_i | d_i$ for $1 \leq i \leq s$ and $\lambda_s(J) \geq b^{u-m}$ we have $R(J; \mathcal{P}) = 0$.

Remark 1. The case $\mathbf{e} = (1, \ldots, 1) \in \mathbb{N}^s$ corresponds to the concept of a (t, m, s)-net in base b with $t = u$. The original definition in [32] has the condition $\lambda_s(J) = b^{u-m}$ instead of $\lambda_s(J) \geq b^{u-m}$. This causes some unusual problems. For instance, a point set $\mathcal{P}$ can be a $(u, m, \mathbf{e}, s)$-net in base b for some u with $0 \leq u < m$, but $\mathcal{P}$ need not be a $(v, m, \mathbf{e}, s)$-net in base b for a certain value of v with $u < v \leq m$ (see the example in [7, Remark 2]). Definition 1 guarantees that a $(u, m, \mathbf{e}, s)$-net in base b is also a $(v, m, \mathbf{e}, s)$-net in base b for any integer v with $u \leq v \leq m$.

The classical notion of a (t, s)-sequence in base b is generalized in Definition 2. As usual, we write $[\mathbf{x}]_{b,m}$ for the coordinatewise m-digit truncation in base b of $\mathbf{x} \in [0, 1]^s$.

Definition 2. Let $b \geq 2$, $s \geq 1$, and $u \geq 0$ be integers and let $\mathbf{e} \in \mathbb{N}^s$. A sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in $[0, 1]^s$ is a *$(u, \mathbf{e}, s)$-sequence in base b* if for all integers $k \geq 0$ and $m > u$ the points $[\mathbf{x}_n]_{b,m}$ with $kb^m \leq n < (k + 1)b^m$ form a $(u, m, \mathbf{e}, s)$-net in base b.

Remark 2. The case $\mathbf{e} = (1, \ldots, 1) \in \mathbb{N}^s$ yields the definition of a (t, s)-sequence in base b with $t = u$. It is obvious that any (u, s)-sequence in base b is a $(u, \mathbf{e}, s)$-sequence in base b for any choice of $\mathbf{e} \in \mathbb{N}^s$.