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Note

# A note on independent vertex–edge domination in graphs☆

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ABSTRACT

The independent vertex–edge domination number and the upper non-enclaving number of a graph $G$ are denoted by $i_{ve}(G)$ and $\Psi(G)$, respectively. Boutrig et al. posed the following question: Let $G$ be a connected graph with order $n$. Is $\Psi(G)+i_{ve}(G)\le n$?

In this paper, we provide an infinite family of counterexamples. A new relationship between $\Psi(G)$ and $i_{ve}(G)$ is established. Furthermore, if $G$ is a connected cubic graph, we answer this question in the affirmative.



## 1. Introduction

Graph theory terminology not presented here can be found in [1]. Let $G=(V,E)$ be a graph with $|V|=n$. The neighborhood and closed neighborhood of a vertex $v$ in the graph $G$ are denoted by $N(v)$ and $N[v]=N(v)\cup\{v\}$, respectively. We denote the *degree* of a vertex $v$ in $G$ by $d(v)=|N(v)|$. Let $\Delta(G)$ denote the maximum degree of $G$. The graph induced by $S\subseteq V$ is denoted by $G[S]$. Let $G-S$ denote the induced subgraph $G[V-S]$.

A set $S$ of vertices in a graph $G$ is a *dominating set* if every vertex not in $S$ is adjacent to a vertex in $S$. If, in addition, $S$ is an independent set, then $S$ is an *independent dominating set*. The *domination number* $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set in $G$, while the *independent domination number* $i(G)$ of $G$ is the minimum cardinality of an independent dominating set in $G$.

A set $S\subseteq V$ is called *irredundant* if for every vertex $v\in S$, $N[v]-N[S-\{v\}]\neq\emptyset$. The maximum cardinality of an irredundant set is called the *upper irredundance number* of $G$ and is denoted $IR(G)$.

A vertex $v\in S$ is called an *enclave* in $S$ if $N[v]\subseteq S$. A set $S$ containing no enclave is called a *non-enclaving set*. The maximum cardinality of a non-enclaving set is called the *upper non-enclaving number* of $G$ and is denoted $\Psi(G)$.

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A vertex $u \in V$ is said to *ve-dominate* an edge $vw \in E$ if

1. $u = v$ or $u = w$, that is, $u$ is incident to $vw$, or
2. $uv$ or $uw$ is an edge in $G$, that is, $u$ is incident to an edge that is adjacent to $vw$.

In other words, a vertex $u$ *ve*-dominates all edges incident to any vertex in $N[u]$. A set $S \subseteq V$ is a *vertex–edge dominating set* (or simply a *ve*-dominating set) if for every edge $e \in E$, there exists a vertex $v \in S$ such that $v$ *ve*-dominates $e$. A *ve*-dominating set $S$ is minimal if, for every vertex $v \in S$, $S \setminus \{v\}$ is not a *ve*-dominating set in $G$. The *vertex–edge domination number*, $\gamma_{ve}(G)$, of $G$ is the minimum cardinality of a vertex–edge dominating set. A set $S \subseteq V$ is an *independent vertex–edge dominating set* (or simply an independent *ve*-dominating set) if $S$ is both independent and *ve*-dominating. The *independent vertex–edge domination number*, $i_{ve}(G)$, of $G$ is the minimum cardinality of an independent *ve*-dominating set.

For any parameter $\mu(G)$ associated to a graph Property $P$, we refer to a set of vertices with Property $P$ and cardinality $\mu(G)$ as a $\mu(G)$-set.

The majority of graph theory research on parameters involved with domination, independence, and irredundance has focused on either sets of vertices or sets of edges; for example, sets of vertices that dominate all other vertices or sets of edges that dominate all other edges. There has been very little research on mixing vertices and edges. An example where vertices and edges "mix" is a total covering. In the classical definition of covering, a vertex (edge) is said to cover all of its incident edges (vertices). In 1977, Alavi et al. [2] introduced a new invariant for both coverings and matchings, which they called total coverings and total matchings. For these invariants, a vertex covers itself, all adjacent vertices, and all incident edges. Also, an edge $e = uv$ covers itself, all adjacent edges, and its two incident vertices $u$ and $v$.

In Chapter 4 of his Ph.D. thesis [3], Peters introduced two related concepts: vertex–edge domination and edge–vertex domination. Traditional (vertex–vertex and edge–edge) domination has been extensively studied and has been surveyed in [4,5]. But the vertex–edge and edge–vertex domination parameters have received very little attention since their introduction over twenty years ago. In 2007, Lewis [6] continued to study these extensions of the definition of domination. He established the vertex–edge domination chain and edge–vertex domination chain. These chains have generated a considerable amount of interest among graph theory researchers.

In 2014, Vijayan et al. [7] studied the concept of vertex–edge domination polynomial of wheels. Vertex–edge domination was also studied in [8,9].

In 2007, Lewis raised the following question.

**Question 1.1** (*Lewis [6]*). *Let $G$ be a connected graph with order $n$. Is $IR(G) + \gamma_{ve}(G) \leq n$?*

It is obvious that $\gamma_{ve}(G) \leq i_{ve}(G)$. Boutrig et al. [1] answered the question in the affirmative. They gave the following result.

**Theorem 1.1** (*Boutrig et al. [1]*). *Let $G$ be a connected graph of order $n$. Then $IR(G) + i_{ve}(G) \leq n$, with equality if and only if $G$ is a star.*

It is known that for any graph $G$, $IR(G) \leq \Psi(G)$. They posed the following open question.

**Question 1.2.** *Let $G$ be a connected graph with order $n$. Is $\Psi(G) + i_{ve}(G) \leq n$?*

In this paper, we construct an infinite family of connected graphs and give a negative answer to the question. A new relationship between $\Psi(G)$ and $i_{ve}(G)$ is established. Furthermore, if $G$ is a connected cubic graph, we answer this question in the affirmative.