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Inheritance of convexity for partition restricted games

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ABSTRACT

A correspondence $\mathcal{P}$ associates to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A and to every game (N, v), the $\mathcal{P}$-restricted game $(N, \overline{v})$ defined by $\overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F)$ for all $A \subseteq N$. We give necessary and sufficient conditions on $\mathcal{P}$ to have inheritance of convexity from (N, v) to $(N, \overline{v})$. The main condition is a cyclic intersecting sequence free condition. As a consequence, we only need to verify inheritance of convexity for unanimity games and for the small class of extremal convex games (N, v_S) (for any $\emptyset \neq S \subseteq N$) defined for any $A \subseteq N$ by $v_S(A) = |A \cap S| - 1$ if $A \cap S \neq \emptyset$, and $v_S(A) = 0$ otherwise. In particular, when $(N, \overline{v})$ corresponds to Myerson's network-restricted game, inheritance of convexity can be verified by this way. For the $\mathcal{P}_{\min}$ correspondence ($\mathcal{P}_{\min}(A)$ is built by deleting edges of minimum weight in the subgraph G_A of a weighted communication graph G), we show that inheritance of convexity for unanimity games already implies inheritance of convexity.

1. Introduction

We consider, on a given finite set N, with $|N| = n$, an arbitrary correspondence $\mathcal{P}$ which associates to every subset $A \subseteq N$ a partition $\mathcal{P}(A)$ of A. Then, for every game (N, v), we define the restricted game $(N, \overline{v})$ associated with $\mathcal{P}$ by:

$$\overline{v}(A) = \sum_{F \in \mathcal{P}(A)} v(F), \quad \text{for all } A \subseteq N. \tag{1}$$

We will more simply refer to this game as the $\mathcal{P}$-restricted game. v is the characteristic function of the game, $v : 2^N \to \mathbb{R}$, $A \mapsto v(A)$ and satisfies $v(\emptyset) = 0$. Through many concrete choices for the correspondence $\mathcal{P}$, the new game $(N, \overline{v})$ can take into account many combinatorial structures and different aspects of cooperation restrictions.

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The first founding example is the Myerson's correspondence $\mathcal{P}_M$ associated with communication games [1]. Communication games are cooperative games (N, v) defined on the set of vertices N of an undirected graph $G = (N, E)$, where E is the set of edges. For every coalition $A \subseteq N$, we consider the induced graph $G_A := (A, E(A))$, where $E(A)$ is the set of edges of E with ends in A. $\mathcal{P}_M(A)$ is the set of connected components of G_A. The $\mathcal{P}_M$-restricted game $(N, \overline{v})$, known as Myerson's game, takes into account how the players of N can communicate according to the graph G. Many other correspondences have been considered to define restricted games (see, e.g., [2–7]).

Of course, for applications as well as for theoretical reasons, it is of major interest to compare the properties of the games (N, v) and $(N, \overline{v})$ and at first to decide if we have inheritance of basic properties as superadditivity and convexity from the underlying game (N, v) to the restricted game $(N, \overline{v})$. In this case, we will say we have inheritance of convexity (resp. superadditivity) for the correspondence $\mathcal{P}$. Inheritance of convexity is a nice property as it implies that good properties are inherited, for instance the non-emptiness of the core, and that the Shapley value is in the core.

Let us observe that inheritance of convexity for a correspondence is a strong property, hence it would be useful to consider weaker properties restricting the inheritance to a smaller class of convex games as, for instance, the class of unanimity games (N, u_S). It is also a first key step to prove inheritance in the general case. In a preceding paper [6], we have established necessary and sufficient conditions on $\mathcal{P}$ to have inheritance of superadditivity on one hand:

$$\text{for all } A \subset B \subseteq N, \quad \text{every block of } \mathcal{P}(A) \text{ is contained in a block of } \mathcal{P}(B), \tag{2}$$

and of convexity for unanimity games on the other hand:

$$\text{for all } A, B \subseteq N, \quad \mathcal{P}(A \cap B) = \{C \in 2^N \setminus \{\emptyset\} \mid \exists F \in \mathcal{P}(A), \exists G \in \mathcal{P}(B) : C = F \cap G\}. \tag{3}$$

Henceforth, assuming that these two last conditions are realized, we introduce, in the present paper, for arbitrary subsets $A, B \subseteq N$ and for every $D \in \mathcal{P}(A \cup B)$, the family of intersecting sequences $\{C_1, C_2, \ldots, C_l\}$, such that $C_j \subseteq D$, $C_j \in \mathcal{P}(A)$ or $C_j \in \mathcal{P}(B)$, for all j, $1 \leq j \leq l$, $C_j \cap C_{j+1} \neq \emptyset$, for all j, $1 \leq j \leq l-1$, $C_1 \setminus C_2 \neq \emptyset$, and $C_l \setminus C_{l-1} \neq \emptyset$. (This definition is close to the definitions of intersecting subsets and intersecting family in [8] or [9].) If $C_1 = C_l$, we call it a cyclic intersecting sequence of $\mathcal{P}$. We say that $\mathcal{P}$ is cyclic intersecting sequence free if such cyclic intersecting sequence does not exist in $\mathcal{P}$.

The main result of this paper is that, assuming (2) and (3) satisfied, we have inheritance of convexity for $\mathcal{P}$ if and only if $\mathcal{P}$ is a cyclic intersecting sequence free correspondence (Theorem 16). We also prove in Theorem 19 that it is enough to verify this condition for subsets $A, B \subseteq N$ such that $|A \setminus B| = |B \setminus A| = 1$. In all proofs we extensively use writing any game as a unique linear combination of unanimity games. To prove that the cyclic intersecting sequence free condition is necessary, we have to use the small class of extremal convex games (N, v_S) $(S \subseteq N, |S| \geq 2)$ with $v_S(A) = |A \cap S| - 1$ if $A \cap S \neq \emptyset$, and $v_S(A) = 0$ otherwise. As a consequence, we only have to verify inheritance of convexity for unanimity games and for this last class of extremal convex games (N, v_S) to obtain inheritance for all convex games. It is a surprising and unexpected fact as the class of convex games is much bigger than these two small classes. We prove that the cyclic intersecting sequence free condition is sufficient by computing explicitly the link between the convexity of the two games (N, v) and $(N, \overline{v})$. For $A, B \subseteq N$, we set:

$$\Delta v(A, B) := v(A \cup B) + v(A \cap B) - v(A) - v(B).$$

We show that, for an explicitly given family of subsets $A_j, B_j \subseteq A \cup B$, $1 \leq j \leq p$ (for some p depending on A and B), we have:

$$\Delta \overline{v}(A, B) = \sum_{j=1}^{p} \Delta v(A_j, B_j).$$