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Polynomial kernels for deletion to classes of acyclic digraphs[☆]

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ABSTRACT

We consider the problem to find a set X of vertices (or arcs) with $|X| \leq k$ in a given digraph G such that $D = G - X$ is an acyclic digraph. In its generality, this is Directed Feedback Vertex Set (or Directed Feedback Arc Set); the existence of a polynomial kernel for these problems is a notorious open problem in the field of kernelization, and little progress has been made. In this paper, we consider the vertex deletion problem with an additional restriction on D, namely that D must be an out-forest, an out-tree, or a (directed) pumpkin. Our main results show that for each of the above three restrictions we can obtain a kernel with $k^{O(1)}$ vertices on general digraphs. We also show that the arc deletion problem with the first two restrictions (out-forest and out-tree) can be solved in polynomial time; this contrasts the status of the vertex deletion problem of these restrictions, which is NP-hard even for acyclic digraphs. The arc and vertex deletion problem with the third restriction (pumpkin), however, can be solved in polynomial time for acyclic digraphs, but becomes NP-hard for general digraphs. We believe that the idea of restricting D could yield new insights towards resolving the status of Directed Feedback Vertex Set.

1. Introduction

In this paper, we study the problem of removing a (small) subset of vertices X from a graph G such that the resulting graph $G - X$ is acyclic. On undirected graphs, this translates immediately to the property that $G - X$ is a forest or (if we insist that $G - X$ is connected) a tree. The problem to decide whether a given undirected graph G has a set $X \subseteq V(G)$ of size at most a given integer k such that $G - X$ is a forest or a tree is known as Feedback Vertex Set and Tree Deletion Set respectively.

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Over the past years, we have gotten to understand the complexity of Feedback Vertex Set and Tree Deletion Set quite well. Both problems are NP-hard [1,2]. It is long known that the minimization version of Feedback Vertex Set admits a polynomial-time 2-approximation algorithm [3,4] and that Feedback Vertex Set admits a polynomial kernel[1] parameterized by the size k of a minimum feedback vertex set [6–8]. The minimization version of Tree Deletion Set, in contrast, cannot be polynomial-time approximated within a factor $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$ [2], unless P = NP. However, Tree Deletion Set was recently shown to admit a polynomial kernel (when parameterized by k) [9].

The usual way to generalize Feedback Vertex Set and Tree Deletion Set to digraphs is to insist that the resulting digraph has no directed cycle. Indeed, the problem to decide whether a given digraph G has a set $X \subseteq V(G)$ of size at most a given integer k such that $G - X$ is a (weakly connected) acyclic digraph is known as Directed Feedback Vertex Set (Connected DAG Vertex Deletion Set). In contrast to their undirected counterparts, the complexity situations for Directed Feedback Vertex Set and Connected DAG Vertex Deletion Set are very much unclear.

It is known that Directed Feedback Vertex Set is NP-hard [1], even on tournaments [10]. Connected DAG Vertex Deletion Set is NP-hard and cannot be polynomial-time approximated within a factor $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$ [2], but we are not aware of any results on the parameterized complexity of this problem. Directed Feedback Vertex Set is polynomial-time approximable within a factor of $O(\log |V(G)| \log\log |V(G)|)$ on general digraphs [11,12], but it is open whether the problem admits a constant-factor approximation in polynomial time. Directed Feedback Vertex Set has a kernel of exponential size $k^{O(k)}$, as the problem was shown fixed-parameter tractable by Chen et al. [13], but it is unknown whether a polynomial kernel exists. In fact, this question remains open despite being posed several times [13–17].

There is limited insight into whether Directed Feedback Vertex Set could admit a polynomial kernel. Abu-Khzam [18] and Dom et al. [19] showed that Directed Feedback Vertex Set admits a polynomial kernel if the given digraph is a (bipartite) tournament and Bang-Jensen et al. [20] recently extended this to generalizations of tournaments. We are not aware of polynomial kernels for Directed Feedback Vertex Set on other restricted classes of digraphs. This suggests to explore other roads towards an answer to the open question of a polynomial kernel for Directed Feedback Vertex Set.

Our contributions We study a different translation of Feedback Vertex Set and Tree Deletion Set to digraphs. Instead of transferring the property that the resulting graph should be acyclic to digraphs, we transfer the property that the resulting graph should be a forest or tree. To this end, we consider the notion of an *out-tree*, which is a digraph where each vertex has in-degree at most 1 and the underlying (undirected) graph is a tree. An *out-forest* is a disjoint union of out-trees. This leads to the following parameterized problems:

Out-Forest/Out-Tree Vertex Deletion Set	
Input:	A digraph G and an integer k.
Question:	Is there a set $X \subseteq V(G)$ with $\lvert X\rvert \le k$ such that $G - X$ is an out-forest/out-tree?

Note that these problems can also be viewed as a restricted version of Directed Feedback Vertex Set and Connected DAG Vertex Deletion Set. Here, instead of restricting the input graph G as Abu-Khzam [18] and Dom et al. [19] did when they considered tournaments, we consider general digraphs G as input but restrict what kind of acyclic digraph the resulting digraph $G - X$ should be.

Thinking further in this direction, we consider another restriction on the resulting digraph, namely that it should be a pumpkin. A digraph is a *pumpkin* if it consists of a source vertex s and a sink vertex t ($s \neq t$),

[1] We refer to the monograph by Cygan et al. [5] for background on parameterized complexity and kernelization.