Discrete Optimization 23 (2017) 33–55

Contents lists available at ScienceDirect

Discrete Optimization

www.elsevier.com/locate/disopt

The separation problem of rounded capacity inequalities: Some polynomial cases

Ibrahima Diarrassouba

Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, 76600 Le Havre, France

ARTICLE INFO

Article history:
Received 16 May 2016
Received in revised form 16 September 2016
Accepted 30 November 2016
Available online 26 December 2016

Keywords:
Polynomial separation algorithm
Rounded capacity inequalities
Capacitated vehicle routing problem
Submodular function

ABSTRACT

In this paper we are interested in the separation problem of the so-called rounded capacity inequalities which are involved in the two-index formulation of the CVRP (Capacitated Vehicle Routing Problem) polytope. In a recent work (Diarrassouba, 2015, [1]), we have investigated the theoretical complexity of that problem in the general case. Several authors have devised heuristic separation algorithms for rounded capacity inequalities for solving the CVRP. In this paper, we investigate the conditions under which this separation problem can be solved in polynomial time, and this, in the context of the CVRP or for solving other combinatorial optimization problems in which rounded capacity inequalities are involved. We present four cases in which they can be separated in polynomial time and reduce the problem to $O(n^2)$ maximum flow computations.

1. Introduction

The Capacitated Vehicle Routing Problem (CVRP) is defined by an undirected complete graph $G = (V, E)$, a special node $s \in V$, called *the depot*, and $V \setminus \{s\}$ the set of clients. Each client $u \in V \setminus \{s\}$ is given a positive demand d_u. To satisfy the demands, we use a fleet of identical vehicles, each of capacity $Q \geq 2$. Every edge $\{u, v\}$ of the graph is assigned a positive routing cost c_{uv}. Each client must be served by a single vehicle and no vehicle can serve a set of clients whose total demand exceeds its capacity. Also, each vehicle must leave and return back to the depot. The problem is to find a set of vehicles and a routing for each vehicle such that all the clients are served and the total routing cost is minimum. The client demands d_u, $u \in V \setminus \{s\}$, and the vehicle capacity are assumed, w.l.o.g, to be positive integers. We also assume that $d_u \leq Q$, for all $u \in V \setminus \{s\}$.

The CVRP is known to be strongly NP-hard, since it generalizes the *Traveling Salesman Problem* and has many practical applications, including network design, routing and scheduling problems. The problem, and its variants, have received a lot of attention for several decades and still remain subject to extensive

E-mail address: diarrasi@univ-lehavre.fr.

http://dx.doi.org/10.1016/j.disopt.2016.11.005
1572-5286/

researches. Several formulations have been proposed for the problem (see for instance [2–6]). In this paper, we are interested in the two-index formulation of the CVRP and the associated capacity inequalities which are defined below.

Given a node set $W \subsetneq V$, the *cut* induced by W in the graph G is denoted by $\delta(W)$ and is the set of edges of G having one node in W and the other in $V \setminus W$. When $W = \{u\}$, we denote by $\delta(u)$ the cut induced by W. Also, for a node set $W \subseteq V \setminus \{s\}$, we let $BP(W)$ be the minimum number of vehicles needed to serve the client set W. Namely, $BP(W)$ is the optimal solution of a *Bin Packing Problem* where bins are of capacity Q, object set is W and each object $u \in W$ is of size d_u.

The CVRP can be formulated by the following integer linear program (see [3,5,6]). Let $x \in \mathbb{R}^E$ be an integer vector, where x_{uv} is the number of times the edge $\{u, v\} \in E$ is used by a vehicle in a solution of the CVRP. For every edge set $F \subseteq E$, we let $x(F) = \sum_{e \in F} x_e$. The *two-index formulation* for the CVRP is given by

$$\min \quad \sum_{e \in E} c_e x_e$$

s.t.

$$x(\delta(u)) = 2, \quad \text{for all } u \in V \setminus \{s\}, \tag{1.1}$$

$$x(\delta(W)) \geq 2BP(W), \quad \text{for all } W \subseteq V \setminus \{s\},\ |W| \geq 2, \tag{1.2}$$

$$0 \leq x_{uv} \leq 1, \quad \text{for all } u, v \in V \setminus \{s\}, \tag{1.3}$$

$$0 \leq x_{su} \leq 2, \quad \text{for all } u \in V \setminus \{s\}, \tag{1.4}$$

$$x_{uv} \in \{0, 1\}, \quad \text{for all } u, v \in V \setminus \{s\}, \tag{1.5}$$

$$x_{su} \in \{0, 1, 2\}, \quad \text{for all } u \in V \setminus \{s\}. \tag{1.6}$$

Constraints (1.1) are the *degree constraints* and indicate that every client is served exactly once. Inequalities (1.2) are the so-called *capacity inequalities*. They ensure that the total demand of a set of clients does not exceed the capacity of the vehicle used to serve these clients. They also ensure that the routes of the vehicles are connected. The polytope described by constraints (1.1)–(1.6) is the so-called *CVRP polytope*, and is denoted by $\text{CVRP}(G, d, Q)$. We also denote by $\text{CVRP}'(G, d, Q)$ the polytope defined by the linear relaxation of the above formulation.

Some variants of the CVRP impose that the number of vehicles used in a solution is bounded by a positive integer k. In this case, the constraint

$$x(\delta(s)) \leq 2k, \tag{1.7}$$

is added to the formulation of the problem. In some other variants, it is required that the number of vehicles used in a solution is exactly k. The inequality in (1.7) hence is replaced by an equality.

In [7], Archetti et al. investigated a variant of the CVRP, called Skip Delivery Problem (SDP for short), in which the vehicle capacity is $Q = 2$ and the demand of the clients may be greater than the vehicle capacity. They [7] showed that the SDP can be solved in polynomial time. Since the CVRP when $Q = 2$ is a particular case of the SDP, the CVRP can be solved in polynomial time in this particular case.

Several authors devised Branch-and-Cut algorithms based on the two-index formulation to solve the CVRP in the general case (see for instance [8,9,5,6]). Since in this formulation, inequalities (1.2) are exponential in number, one has to address the separation problem associated with $\text{CVRP}'(G, d, Q)$, in order to solve the linear relaxation of the CVRP using a cutting plane method. Recall that the *separation problem* associated with a polyhedron P is to say if a given solution $\overline{x}$ belongs to P or not, and if $\overline{x} \notin P$, then exhibit an inequality $ax \geq \alpha$ valid for P and violated by $\overline{x}$. We also define the separation problem associated with a solution $\overline{x}$ and a family $\mathcal{F}$ of inequalities, which consists in saying if $\overline{x}$ satisfies or not all the inequalities of $\mathcal{F}$, and if not, exhibit at least one inequality of $\mathcal{F}$ which is violated by $\overline{x}$. An algorithm