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On the k-limited packing numbers in graphs

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ABSTRACT

We give a sharp lower bound on the lower k-limited packing number of a general graph. Moreover, we establish a Nordhaus–Gaddum type bound on 2-limited packing number of a graph G of order n as $L_2(G) + L_2(\bar{G}) \le n + 2$. Also, we investigate the concepts of packing number (1-limited packing number) and open packing number in graphs with more details. In this way, by making use of the well-known result of Farber (1984) for strongly chordal graphs and its total version (2005) for trees we prove the new upper bound $\gamma(G) \le (n - \ell + \delta' s)/(1 + \delta')$ for every connected strongly chordal graph G of order $n \ge 3$ with ℓ pendant vertices and s support vertices, where δ' is the minimum degree taken over all vertices that are not pendant vertices, and improve $\gamma_t(T) \le (n + s)/2$ for every tree T, that was first proved by Chellali and Haynes in 2004.

1. Introduction

Throughout this paper, let G be a finite connected graph with vertex set $V = V(G)$, edge set $E = E(G)$, minimum degree $\delta = \delta(G)$ and maximum degree $\Delta = \Delta(G)$. Recall that a *pendant vertex* of G (a *leaf* of a tree T) is a vertex of degree 1 and a support vertex is a vertex having at least one pendant vertex in its neighborhood. We use [1] as a reference for terminology and notation which are not defined here. For any vertex $v \in V$, $N(v) = \{u \in G \mid uv \in E(G)\}$ denotes the *open neighborhood* of v in G, and $N[v] = N(v) \cup \{v\}$ denotes its *closed neighborhood*. For a set A of vertices, its open neighborhood is the set $N(A) = \bigcup_{v \in A} N(v)$. The complement $\bar{G}$ of a graph G has vertex set V and $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$. For any graph parameter ψ, bounds on $\psi(G) + \psi(\bar{G})$ and $\psi(G)\psi(\bar{G})$ are called Nordhaus–Gaddum inequalities. For more information about this subject the reader can consult [2].

A set $S \subseteq V$ is a *dominating set (total dominating set)* in G if each vertex in $V \setminus S$ (in V) is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ *(total domination number* $\gamma_t(G)$*)* is the minimum cardinality of a dominating set (total dominating set) in G. A subset $B \subseteq V$ is a *packing (open packing)* in

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G if for every distinct vertices $u, v \in B$, $N[u] \cap N[v] = \emptyset$ $(N(u) \cap N(v) = \emptyset)$. The *packing number (open packing number)* $\rho(G)$ $(\rho^o(G))$ is the maximum cardinality of a packing (an open packing) in G. Also, the *lower packing number*, denoted $\rho_L(G)$, is the minimum cardinality of a maximal packing in G.

Clearly, $B \subseteq V$ is a packing (an open packing) in G if and only if $|N[v] \cap B| \leq 1$ $(|N(v) \cap B| \leq 1)$, for all $v \in V$. Here, we prefer to work with these definitions on these parameters rather than the previous ones.

Gallant et al. [3] introduced the concept of limited packing in graphs and exhibited some real-world applications of it to network security, market saturation and codes (also, the authors in [4] presented some results as an application of the concept of the limited packing). A subset $B \subseteq V$ is a *k-limited packing* in G if $|N[u] \cap B| \leq k$, for every vertex $u \in V$. The *k-limited packing number* $L_k(G)$ is the maximum cardinality of a k-limited packing in G. Also, the *lower k-limited packing number*, denoted $L_k^\ell(G)$, is the minimum cardinality of a maximal k-limited packing in G. Obviously, $L_k^\ell(G) \leq L_k(G)$. These two concepts generalize the concepts of packing and lower packing, respectively.

We prove a sharp lower bound on lower k-limited packing number of a general graph G and drive the lower bound $n/2$ for lower 2-limited packing number of a cubic graph of order n that is stronger than its similar result in [3]. Also, we show that $L_2(G) + L_2(\bar{G}) \leq n + 2$ for each graph G of order n.

Farber [5] proved that $\rho(G) = \gamma(G)$, for every strongly chordal graph G. We prove an upper bound on packing number of a graph and use this well-known result to show that $(n - \ell + \delta' s)/(1 + \delta')$ is an upper bound on domination number of a connected strongly chordal graph of order $n \geq 3$ with ℓ pendant vertices and s support vertices, where δ' is the minimum degree taken over all vertices that are not pendant vertices. Finally we improve $\gamma_t(T) \leq (n + s)/2$ for every tree T, that was first proved by Chellali and Haynes [6], as an immediate result.

2. Results on the k-limited packing number

Gallant et al. [3] exhibited the lower bound $L_2(G) \geq n/4$, for a cubic graph G of order n. In this section we establish a tight lower bound on lower k-limited packing number of a general graph, also we observe that it slightly improves the lower bound in [3] as the special case $k = 2$. First, we need the following useful lemma that is an immediate result of the definition of a maximal k-limited packing.

Lemma 2.1. *Let G be a graph and k be a positive integer. Then a k-limited packing set in G is maximal if and only if every vertex in $V \setminus B$ belongs to the closed neighborhood of a vertex u with $|N[u] \cap B| = k$.*

We are now in a position to present the main theorem of this section.

Theorem 2.2. *If G is graph of order n and k is a positive integer, then*

$$L_k^\ell(G) \geq \frac{kn}{\Delta(\Delta - k + 1) + k}$$

and this bound is sharp.

Proof. Since $L_k^\ell(G) = n$ for $k \geq \Delta + 1$, we may assume that $k \leq \Delta$. Let B be a maximal k-limited packing set in G such that $|B| = L_k^\ell(G)$. For all $i \leq k \leq \Delta$, we define

$$B_i = \{u \in B \,|\, |N[u] \cap B| = i\} \quad \& \quad B'_i = \{u \in V \setminus B \,|\, |N[u] \cap B| = i\}.$$

Obviously, $|B| = \sum_{i=1}^{k} |B_i|$ and $|V \setminus B| = \sum_{i=0}^{k} |B'_i|$.

By Lemma 2.1, every vertex in B'_0 has at least one neighbor in B'_k. On the other hand, every vertex in B'_k has at most $\Delta - k$ neighbors in B'_0. It follows that

$$|B'_0| \leq |[B'_0, B'_k]| \leq (\Delta - k)|B'_k|$$