Discrete Optimization 22 (2016) 372–388

Contents lists available at ScienceDirect

Discrete Optimization

www.elsevier.com/locate/disopt

Constrained domatic bipartition on trees

Giovanni Andreatta[a], Carla De Francesco[a,*], Luigi De Giovanni[a], Paolo Serafini[b]

[a] *Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy*

[b] *Dipartimento di Scienze Matematiche, Informatiche e Fisiche, Università di Udine, viale delle Scienze 206, 33100 Udine, Italy*

ARTICLE INFO

Article history:
Received 4 September 2015
Received in revised form 20 May 2016
Accepted 21 October 2016
Available online 10 November 2016

MSC:
05C69
90C10
90C27
90C57
90C35
90C39

Keywords:
Dominating set
Cardinality constraint
Tree
Polynomial-time algorithm
Integral polytope

ABSTRACT

Given an undirected graph, the Constrained Domatic Bipartition Problem (CDBP) consists in determining a bipartition, if it exists, of the nodes into two dominating sets, with the additional constraint that one of the two subsets has a given cardinality. The problem is NP-hard in general and in this paper we focus on trees. First, we provide explicit solutions in simple cases, i.e., stars and paths. Then, we provide a polyhedral representation for all domatic bipartitions of a tree. Although the matrix associated with the polyhedron is not totally unimodular, we prove that all its vertices have integral components. Adding the cardinality constraint, the resulting polyhedron will generally lose this property. We then propose a constructive, dynamic programming algorithm for CDBP on trees, that is able to simultaneously find a solution for all possible cardinalities. The proposed algorithm is polynomial with complexity $O(n^3)$, where n is the number of nodes. Finally, we discuss the extension of CDBP to the weighted case, show that it is NP-hard and provide a pseudo-polynomial algorithm for the problem.

1. Introduction

Given an undirected graph $G = (V, E)$, a subset of nodes is *dominating* if each node of G is either in the subset or is adjacent to some node in the subset. A *domatic partition* is a partition of V into dominating sets.

The problem of splitting V into 2 dominating sets has always a positive answer, assuming G has no isolated nodes [1]. In fact, if the graph is a tree then it is connected and bipartite and the two subsets of nodes in the bipartition give two dominating sets [2]. Otherwise, it suffices to consider a spanning tree for each connected component.

* Corresponding author.
E-mail addresses: giovanni@math.unipd.it (G. Andreatta), carla@math.unipd.it (C. De Francesco), luigi@math.unipd.it (L. De Giovanni), paolo.serafini@uniud.it (P. Serafini).

http://dx.doi.org/10.1016/j.disopt.2016.10.003
1572-5286/

By constraining the cardinalities of the two sets we have the following problem, that we call Constrained Domatic Bipartition Problem (CDBP): given a graph $G = (V, E)$ and a number p with $1 \leq p < |V|$, determine a bipartition, if it exists, of V into 2 dominating sets V' and V'' such that $|V'| = p$.

This problem has been brought to our attention while working on a problem of splitting the node set of a graph into subsets such that, roughly speaking, all subsets are as close as possible to each other. We fully investigated this problem in [3], but the computational complexity of the case with two balanced sets of nodes was left as an open question. This case is closely connected to CDBP and, to the best of our knowledge, the computational complexity of CDBP has not been investigated in the literature.

In a separate paper [4] we show that CDBP is NP-hard in general and in this paper we show that it is polynomial on trees by providing a dynamic programming algorithm which produces a bipartition for any value of p, if it exists.

We also investigate CDBP from a polyhedral point of view. We show that the domatic bipartition polyhedron (i.e., the polyhedron whose vertices are incidence vectors of one of the two dominating sets) can be represented by a set of $O(|V|)$ linear inequalities in $O(|V|)$ variables. In fact, the vertices associated to this representation have binary components, even if the inequalities do not correspond to a totally unimodular matrix. This allows to find a domatic bipartition with minimum p by solving a linear programming problem. It is already known that finding a domatic bipartition on a tree with the minimum value of p can be done in linear time by using the algorithm proposed in [5,2]. However, adding a cardinality constraint may introduce fractional vertices and therefore linear programming cannot be used to solve CDBP. The polynomial algorithm mentioned above can be used in this case.

The paper is structured as follows. In Section 2 we recall the H-Domatic Partition problem, the H-Shift Coloring problem and the m-node Domatic Partition problem. We discuss some similarities with CDBP and highlight the important differences. In Section 3 we preliminarily consider special classes of trees, namely, stars and paths, for which the solution is straightforward, and simple necessary and sufficient conditions can be given on p for CDBP to have a solution. In Section 4 we provide a polyhedral representation of all domatic bipartitions of a tree. The dynamic programming algorithm to solve CDBP is described in Section 5. Finally, Section 6 discusses the extension of CDBP to the weighted case, shows that it is NP-hard on trees and provides a pseudo-polynomial algorithm for the problem.

2. Connections with existing literature

Given G and an integer H, the H-Domatic Partition Problem asks whether there exists a partition of V into H dominating subsets. The optimization version of the problem gives rise to the Domatic Number Problem, where the maximum H (called *domatic number*) has to be found, and the Domatic Partition Problem, where a domatic partition having a maximum H has to be determined. Problems related to the domatic partition of a graph have been the object of several studies, showing, among other results, that the H-Domatic Partition Problem is NP-complete for $H \geq 3$ [6], while determining the domatic number is polynomial on some special classes of graphs ([2,7,8] among others). The H-Domatic Partition Problem with $H = 2$ has always a positive answer, assuming G has no isolated nodes [1].

The H-Domatic Partition Problem has also relations with the H-Shift Coloring Problem [3]: given a graph $G = (V, E)$ (with edges of arbitrary length and no weights on the nodes) and an integer H, we want to determine a partition of the nodes into H subsets $V_1, V_2, \ldots, V_H$ such that the total distance $D_{tot} = \sum_{v \in V} \sum_{i=1}^{H} d(v; V_i)$ is minimized, where $d(v; U) = \min\{d(v; u) : u \in U\}$ and $d(v; u)$ is the length of a shortest path between v and u. The problem is NP-hard in general [3] and polynomial on trees [9]. When all distances are unitary, then an optimal H-shift coloring is an H-domatic partition if and only if $D_{tot} = |V|(H - 1)$.

A problem related to CDBP is the m-node domatic partition problem [10], where one has to find the maximum number of pairwise disjoint dominating sets whose union has cardinality at most m. Since the