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The minimum weakly connected independent set problem: Polyhedral results and Branch–and–Cut

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ABSTRACT

Let $G = (V, E)$ be a connected graph. An independent set $W \subset V$ is said to be weakly connected if the spanning subgraph $G_W = (V, \delta(W))$ is connected where $\delta(W)$ is the set of edges with exactly one end in W. We present an integer programming formulation for the minimum weakly connected independent set problem and discuss its associated polytope. Some classical graph operations are also used for defining new polyhedra from pieces. We give valid inequalities and describe heuristic separation procedures. Finally, we develop a Branch-and-Cut algorithm and test it on two classes of graphs.

1. Introduction

A wireless sensor network (*WSN*) generally consists of a set of autonomous components which collect data like temperature or pressure and broadcast messages to a base station. The communications between two sensors can be achieved via a shared bandwidth directly if the devices are close enough or through relays provided by intermediary sensors. Unfortunately, the network performance is reduced by interferences since sensors are generally deployed on difficult access sites and unavoidable retransmissions can increase energy consumption. Thus, connectivity is one of the main objectives to be optimized in the design of wireless sensor networks. As there is no physical infrastructure like in wired networks, *a virtual backbone* needs to be created. An efficient hierarchical organization is obtained by dividing the set of sensors into clusters [1]. Each cluster is controlled by a particular node called *clusterhead*. These control nodes are selected to gather information from their neighboring nodes and to communicate with each other.

An undirected communication graph $G = (V, E)$ is usually associated to the sensors located in the region they monitor. The node set V is the set of sensors and an edge $e = (u, v)$ in E corresponds to a possible

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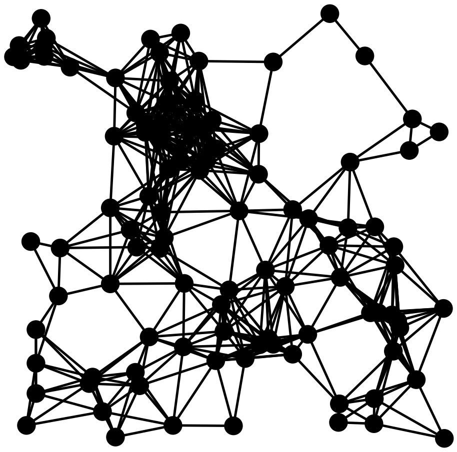

Fig. 1. A communication graph.

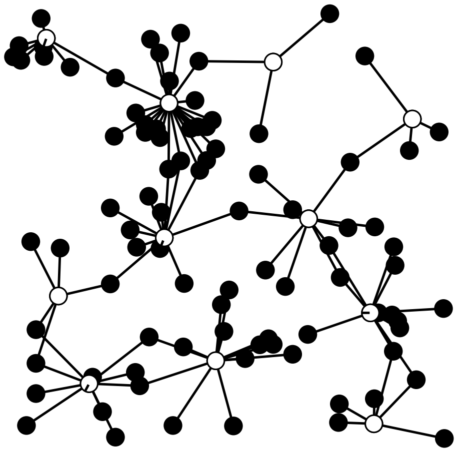

Fig. 2. A virtual backbone.

transmission between two sensors u and v (Fig. 1). This link depends on the euclidian distance between u and v and the energy to deploy for this connection. Generally the size of a *WSN* is large and its nodes have very limited resources. So a virtual backbone (Fig. 2) should be built with low communication and computation costs. The most used structures are based on *dominating* sets and *independent* sets.

Connected Dominating Sets [2] have been proposed as a solution by many authors [3–6]. A node set D is *a connected dominating set*, or *cds* for short, if each vertex in G is in D or adjacent to at least one of the vertices in D (domination property) and if the subgraph induced by D is connected. Thus, communications are ensured between all the vertices via the set D. As one wants to reduce the number of exchanged messages and to avoid useless energy consumption, D must be of small size. But obtaining a minimum connected dominating set is an NP-hard problem [7]. Consequently, the authors in [8,9] used exact algorithms to obtain lower bounds on minimum *cds* and many approximation algorithms and heuristics have been proposed for that problem [10,11]. The *cds* notion can be weakened by using a *weakly connected dominating set*, or *wcds* for short [12,13]. A dominating set D is said to be weakly connected if the partial graph (V, F) is a connected graph, where F is the set of edges of E having at least one end in D. Nevertheless, the problem of minimizing the cardinality of a *wcds* is again NP-hard [7].