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On the mixed set covering, packing and partitioning polytope

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ABSTRACT

We study the polyhedral structure of the mixed set covering, packing and partitioning problem, which has drawn little attention in the literature but has many real-life applications. By considering the "interactions" between the different types of edges of an induced graph, we develop new classes of valid inequalities. In particular, we derive the (generalized) mixed odd hole inequalities, and identify sufficient conditions for them to be facet-defining. In the special case when the induced graph is a mixed odd hole, the inclusion of this new facet-defining inequality provide a complete polyhedral characterization of the mixed odd hole polytope. Computational experiments indicate that these new valid inequalities may be effective in reducing the computation time in solving mixed covering and packing problems.

1. Introduction

We consider an optimization problem with set *partitioning*, *covering* and *packing* constraints, i.e.,

$$\begin{aligned}
(MP):\quad & \max c^T x & & \\
\text{subject to}\quad & A^{=}x = \mathbf{1_{m_0}} \quad (\textbf{partitioning}) & & (1)\\
& A^{\geq}x \geq \mathbf{1_{m_1}} \quad (\textbf{covering}) & & (2)\\
& A^{\leq}x \leq \mathbf{1_{m_2}} \quad (\textbf{packing}) & & (3)\\
& x \in \mathbb{B}^n & &
\end{aligned}$$

where $c \in \mathbb{R}^n, A^{=} \in \mathbb{B}^{m_0 \times n}, A^{\geq} \in \mathbb{B}^{m_1 \times n}, A^{\leq} \in \mathbb{B}^{m_2 \times n}$, $\mathbf{1_m}$ is an all-one vector of dimension m, and $\mathbb{B} = \{0, 1\}$.

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From a practical point of view, the individual covering, packing and partitioning problems, i.e., the problems of which only one class of the inequalities are present, have drawn attention from researchers and practitioners for several decades because of their many real-life applications. Applications of partitioning and covering include staff scheduling [1–4], railroad crew scheduling [5], vehicle routing [6,7] and facility location [8] problems. In these applications, the partitioning (or covering) constraints ensure that each task or customer is covered or served by exactly (or at least) one shift, route or facility. Some areas of application of the set packing problem are train routing [9], auction design [10] and ship scheduling [11]. In these applications, the packing constraints ensure that conflicting assignments are not made. The three individual problems have been separately widely-studied from both theoretical and practical aspects. However, in practice, the three kinds of constraints often appear simultaneously. For example, in multi-period staff-scheduling, the partitioning constraints ensure that exactly one task is assigned to each worker in each time period; the covering constraints guarantee that there are sufficient workers assigned to each task. The packing constraints act to eliminate prohibited co-assignments (e.g., same worker assigned to consecutive night shifts) due to work rules or other considerations.

In this paper, we consider the "interactions" between the different types of constraints and develop new classes of valid inequalities. In particular, we derive the *mixed odd hole inequalities*, and identify sufficient conditions for them to be facet-defining. Our computational experiments indicate that the formulation tightened by these inequalities could reduce the computational effort in solving the mixed problem. Moreover, in the special case when the induced graph of the problem is a mixed odd hole (a chordless odd cycle), the addition of this inequality gives a complete polyhedral characterization of the corresponding polytope. To the best of our knowledge, this is the first paper to completely characterize this mixed odd hole polytope, and introduce new classes of facet-defining inequalities for the general mixed problem.

We refer to problem (MP) with only one class of the constraints (i.e. when only one of m_0, m_1 and m_2 is non-zero) as an *individual problem*, and problem (MP) as the *mixed partitioning, covering and packing problem*. Thus, the NP-hardness of the mixed problem follows directly from the NP-hardness of the individual partitioning, covering or packing problem. Therefore, from a theoretical perspective, it is of great interest to investigate the mixed problem when two or more types of constraints – among (1)–(3) – are present simultaneously.

In essence, a partitioning constraint can be replaced by a covering constraint and a packing constraint. Therefore, in the rest of this paper, W.L.O.G., we focus on the mixed problem with only covering and packing constraints, whose feasible set is:

$$P = \{x \in \mathbb{B}^n : A^{\geq} x \geq \mathbf{1}_{m_1}, A^{\leq} x \leq \mathbf{1}_{m_2}\}.$$

W.L.O.G., we further assume that there is at least one constraint of (2) and one constraint of (3) and that $A^{\geq}$ and $A^{\leq}$ are matrices whose rows form a *clutter*, that is, the support of any row cannot be a subset of the support of another row.

In general, $conv(P)$ is not the intersection of the convex hulls of the associated individual covering and packing polytopes, as shown by the counter-example in the Appendix.

The remainder of our paper is organized as follows. In Section 2, we provide a brief literature review. In Section 3, we investigate the polyhedral structure of the problem for the special case when the constraint matrix induces a graph which we call a mixed odd hole (a chordless odd cycle). We derive the mixed odd hole inequality and show that its inclusion completely characterizes the mixed odd hole polytope. In Section 4, we derive the generalized mixed odd hole inequality and introduce several other classes of valid inequalities for the mixed covering and packing problem. In Section 5, we provide computational results to show the effectiveness of these inequalities in reducing the computational effort to solve mixed covering, packing and partitioning problems.