Operations Research Letters 45 (2017) 575–580

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Robust alternative theorem for linear inequalities with applications to robust multiobjective optimization

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ARTICLE INFO

Article history:
Received 14 April 2017
Received in revised form 4 September 2017
Accepted 4 September 2017
Available online 18 September 2017

Keywords:
Alternative theorem
Linear inequality systems
Linear matrix inequality
Multiobjective optimization
Semi-infinite program

ABSTRACT

We first establish a new alternative theorem for a robust linear inequality system, where the dual statement is expressed in terms of linear matrix inequalities and thus, it can be verified by solving a semidefinite linear program. We then apply the established alternative theorem to derive a characterization of optimality for weakly Pareto solutions of a robust linear multiobjective optimization problem, and to examine weak, strong and converse duality relations in robust linear multiobjective optimization.

1. Introduction

This paper is concerned with an *uncertain linear inequality* system

$$x \in \mathbb{R}^n, \quad a_j(u_j)^\top x \le b_j(u_j), \quad j = 1, \ldots, q, \tag{SU}$$

where $a_j : \mathbb{R}^s \to \mathbb{R}^n, b_j : \mathbb{R}^s \to \mathbb{R}, j = 1, \ldots, q$, are affine mappings given respectively by $a_j(u_j) := a_j^0 + \sum_{i=1}^s u_j^i a_j^i$ and $b_j(u_j) = b_j^0 + \sum_{i=1}^s u_j^i b_j^i$ for $u_j := (u_j^1, \ldots, u_j^s) \in \mathbb{R}^s$ with $a_j^i \in \mathbb{R}^n, b_j^i \in \mathbb{R}, i = 0, 1, \ldots, s, j = 1, \ldots, q$ fixed, and $u_j, j = 1, \ldots, q$, are *uncertain* and they belong to the prescribed *uncertainty* sets $U_j \subset \mathbb{R}^s, j = 1, \ldots, q$.

Following the deterministic approach in robust optimization (see e.g., [1,2]), we investigate the uncertain linear inequality system (SU) by examining its *robust* counterpart:

$$x \in \mathbb{R}^n, \quad a_j(u_j)^\top x \le b_j(u_j), \quad \forall u_j \in U_j, \ j = 1, \ldots, q. \tag{SR}$$

Note that in the robust counterpart (SR) the parameter uncertain $u_j, j = 1, \ldots, q$, are enforced for every possible value of the data within the uncertainty set $U_j, j = 1, \ldots, q$. We refer the interested reader to [5,6] for some characterizations and computations of the radius of robust error bounds for an uncertain linear inequality system with general compact uncertainty sets $U_j \subset \mathbb{R}^{n+1}, j = 1, \ldots, q$.

It is well-known that alternative theorems for finite systems of linear/convex inequalities have played important roles in the development of optimality conditions and duality for continuous optimization problems and in the convergence analysis of optimization algorithms; see e.g., [8,14,15,17,23] and the references therein. Unlike theorems of the alternative for *finite* linear systems and Farkas lemmas (cf. [9,13,14,20,21]), which provide a numerically checkable alternative certificate of the solvability of the given linear system, alternative theorems for *infinite* or *robust* inequality systems do not provide such a certificate in general inasmuch as they relate to arbitrary (even without topological structures) index sets (cf. [13]).

Throughout this paper, the uncertainty sets $U_j, j = 1, \ldots, q$, are assumed to be *compact* and described by *spectrahedrons* (see e.g., [22,24]); that is,

$$U_j := \left\{ u_j := (u_j^1, \ldots, u_j^s) \in \mathbb{R}^s \mid A_j^0 + \sum_{i=1}^s u_j^i A_j^i \succeq 0 \right\}, \quad j = 1, \ldots, q, \tag{1.1}$$

where $A_j^i, i = 0, 1, \ldots, s, j = 1, \ldots, q$, are symmetric $(m_j \times m_j)$ matrices with $m_j \in \mathbb{N} := \{1, 2, \ldots\}$, and the linear matrix inequalities $A_j^0 + \sum_{i=1}^s u_j^i A_j^i \succeq 0, j = 1, \ldots, q$, signify that the matrices $A_j^0 + \sum_{i=1}^s u_j^i A_j^i, j = 1, \ldots, q$, are positive semi-definite. It is worth mentioning here that the spectrahedrons (1.1) possess a large spectrum of infinite convex sets that appear in applications, and they contain commonly used uncertainty sets of robust optimization like ellipsoids, balls, polytopes and boxes [1,2].

The content of this paper is as follows. In Section 2, we establish a new alternative theorem for the *robust* linear inequality system (SR) in which the dual statement can be checked by using a semidefinite linear programming (cf. [7]). This is achieved by

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http://dx.doi.org/10.1016/j.orl.2017.09.002
0167-6377/

means of employing *nice* structures with linear matrix inequality representations of the spectrahedral index sets (1.1) together with a special variable transformation (cf. [7,16]), which paves the way to present the dual statement in terms of linear matrix inequalities. In Section 3, we apply the established alternative theorem to derive a characterization of optimality for weakly Pareto solutions of a *robust linear multiobjective* optimization problem, and to explore weak, strong and converse duality relations in robust linear multi-objective optimization. The reader is referred to [3,11,12] for some duality results on linear/nonsmooth semi-infinite multiobjective optimization problems with arbitrary index sets, and to [4,19] for several results about optimality conditions and/or duality of nonsmooth robust multiobjective optimization problems.

2. Robust alternative theorem for linear inequalities

Throughout the paper, for each $n \in \mathbb{N}$, $\mathbb{R}^n$ signifies the Euclidean space whose norm is denoted by $\|\cdot\|$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. Given a nonempty set $\Omega \subset \mathbb{R}^n$, the topological closure of Ω is denoted by $\mathrm{cl}\,\Omega$, and the convex conical hull of Ω is denoted by $\mathrm{cone}\,\Omega$, i.e., $\mathrm{cone}\,\Omega := \mathbb{R}_+\mathrm{conv}\,\Omega$, where $\mathrm{conv}\,\Omega$ denotes the convex hull of Ω and $\mathbb{R}_+ := [0, +\infty) \subset \mathbb{R}$. A symmetric $(n \times n)$ matrix A is said to be positive semi-definite, denoted by $A \succeq 0$, whenever $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$.

The main result of this paper is stated as follows.

Theorem 2.1 (*Robust Alternative Theorem*). *Let* $(\wp_l, r_l) \in \mathbb{R}^n \times \mathbb{R}, l = 1, \ldots, p$, *and let the cone* $C := \mathrm{cone}\Big\{\big(a_j(u_j), b_j(u_j)\big) \mid u_j \in U_j, j = 1, \ldots, q\Big\}$ *be closed. Assume that the robust linear inequality system* (SR) *has a solution; i.e.,*

$$X := \{x \in \mathbb{R}^n \mid a_j(u_j)^\top x \leq b_j(u_j),\ \forall u_j \in U_j,\ j = 1, \ldots, q\} \neq \emptyset.$$

Then, exactly one the following two statements holds:

(i) $\{x \in X \mid \wp_l^\top x < r_l,\ l = 1, \ldots, p\} \neq \emptyset$;

(ii) $\exists \nu_l \geq 0, l = 1, \ldots, p, \sum_{l=1}^p \nu_l = 1, \exists \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ *such that*

$$\sum_{l=1}^p \nu_l \wp_l + \sum_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i) = 0,$$

$$-\sum_{l=1}^p \nu_l r_l - \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) \geq 0,$$

and $\lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i \succeq 0, j = 1, \ldots, q.$

To prove this theorem, we need the following general version of Farkas's Lemma, which can be found in [14, Theorem 4.3.4].

Lemma 2.2 (*Generalized Farkas*). *Let be given* (b, r) *and* (s_j, p_j) *in* $\mathbb{R}^n \times \mathbb{R}$, *where* j *varies in an arbitrary index set* J. *Suppose that the system of inequalities*

$$s_j^\top x \leq p_j \text{ for all } j \in J \tag{2.1}$$

has a solution $x \in \mathbb{R}^n$. *Then, the following two properties are equivalent:*

(i) $b^\top x \leq r$ *for all* x *satisfying* (2.1)*;*

(ii) $(b, r) \in \mathrm{cl\,cone}\{(0_n, 1) \cup (s_j, p_j) \mid j \in J\}$.

Proof of Theorem 2.1. Let

$$\tilde{C} := \mathrm{cone}\Big\{(0_n, 1) \cup \big(a_j^0 + \sum_{i=1}^s u_j^i a_j^i, b_j^0 + \sum_{i=1}^s u_j^i b_j^i\big) \mid u_j \in U_j, j = 1, \ldots, q\Big\}. \tag{2.2}$$

We first note that the cone $\tilde{C}$ is known as *characteristic cone* (cf. [13]) and it is closed under our assumption as shown in the proof of [7, Theorem 2.1].

[**Not (i)** $\Longrightarrow$ **(ii)**] Assume that (i) fails. Then, let $x \in X$. There exists $l_0 \in \{1, \ldots, p\}$ such that

$$\wp_{l_0}^\top x \geq r_{l_0}.$$

Now, invoking Lemma 2.2, we conclude that

$$(-\wp_{l_0}, -r_{l_0}) \in \mathrm{cl}\tilde{C} = \tilde{C}.$$

Then, there exist $\lambda_0 \geq 0$, $\mu_j \geq 0$, and $u_j := (u_j^1, \ldots, u_j^s) \in U_j, j = 1, \ldots, q$ such that

$$-\wp_{l_0} = \sum_{j=1}^q \mu_j\big(a_j^0 + \sum_{i=1}^s u_j^i a_j^i\big),$$

$$-r_{l_0} = \lambda_0 + \sum_{j=1}^q \mu_j\big(b_j^0 + \sum_{i=1}^s u_j^i b_j^i\big).$$

Putting $\nu_{l_0} := 1$, $\nu_l := 0$ for $l \in \{1, \ldots, p\} \setminus \{l_0\}$, and $\lambda_j^0 := \mu_j \geq 0$, $\lambda_j^i := \mu_j u_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$, we see that $\sum_{l=1}^p \nu_l = 1$ and

$$\sum_{l=1}^p \nu_l \wp_l + \sum_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i) = 0,$$

$$\sum_{l=1}^p \nu_l r_l + \lambda_0 + \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) = 0.$$

The later equality means that $-\sum_{l=1}^p \nu_l r_l - \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) = \lambda_0 \geq 0$.

Consider $j \in \{1, \ldots, q\}$ arbitrary. The relation $u_j \in U_j$ ensures that $A_j^0 + \sum_{i=1}^s u_j^i A_j^i \succeq 0$. We will verify that

$$\lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i \succeq 0. \tag{2.3}$$

Indeed, if $\lambda_j^0 = 0$, then $\lambda_j^i = 0$ for all $i = 1, \ldots, s$, and hence, (2.3) holds trivially. If $\lambda_j^0 \neq 0$, then

$$\lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i = \lambda_j^0 \left(A_j^0 + \sum_{i=1}^s \frac{\lambda_j^i}{\lambda_j^0} A_j^i\right) = \lambda_j^0 \left(A_j^0 + \sum_{i=1}^s u_j^i A_j^i\right) \succeq 0,$$

showing (2.3) holds, too. Consequently, (ii) is valid.

[**(ii)** $\Longrightarrow$ **Not (i)**] Assume that (ii) holds. It means that there exist $\nu_l \geq 0, l = 1, \ldots, p, \sum_{l=1}^p \nu_l = 1, \lambda_j^0 \geq 0, \lambda_j^i \in \mathbb{R}, j = 1, \ldots, q, i = 1, \ldots, s$ such that $\sum_{l=1}^p \nu_l \wp_l + \sum_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i) = 0$, $-\sum_{l=1}^p \nu_l r_l - \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i) \geq 0$ and $\lambda_j^0 A_j^0 + \sum_{i=1}^s \lambda_j^i A_j^i \succeq 0, j = 1, \ldots, q$. By letting $\wp := \sum_{l=1}^p \nu_l \wp_l$, $r := \sum_{l=1}^p \nu_l r_l$, $\lambda_0 := -\sum_{l=1}^p \nu_l r_l - \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i)$, we obtain that $\lambda_0 \geq 0$ and that

$$\begin{aligned} -\wp &= \sum_{j=1}^q (\lambda_j^0 a_j^0 + \sum_{i=1}^s \lambda_j^i a_j^i), \\ -r &= \lambda_0 + \sum_{j=1}^q (\lambda_j^0 b_j^0 + \sum_{i=1}^s \lambda_j^i b_j^i), \end{aligned} \tag{2.4}$$