Operations Research Letters 45 (2017) 325–333

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A convergent hierarchy of SDP relaxations for a class of hard robust global polynomial optimization problems

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ARTICLE INFO

Article history:
Received 29 June 2016
Received in revised form
12 April 2017
Accepted 13 April 2017
Available online 2 May 2017

Keywords:
Robust optimization
Global polynomial optimization
Optimization under data uncertainty
Nonconvex optimization
Semi-definite programming relaxations

ABSTRACT

A hierarchy of semidefinite programming (SDP) relaxations is proposed for solving a broad class of hard nonconvex robust polynomial optimization problems under constraint data uncertainty, described by convex quadratic inequalities. This class of robust polynomial optimization problems, in general, does not admit exact semidefinite program reformulations. Convergence of the proposed SDP hierarchy is given under suitable and easily verifiable conditions. Known exact relaxation results are also deduced from the proposed scheme for the special class of robust convex quadratic programs. Numerical examples are provided, demonstrating the results.

1. Introduction

In this paper we consider the robust polynomial optimization problem,

$$(RP) \quad \min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{subject to} \quad g_i^0(x) + \sum_{j=1}^{s} u_i^j g_i^j(x) \le 0, \quad \forall u_i = (u_i^1,\dots,u_i^s) \in \mathcal{U}_i,\ i = 1,\dots,m,$$

where the objective function f and the constraint functions g_i^j are all polynomials on $\mathbb{R}^n$. Here, $u_i = (u_i^1,\dots,u_i^s)$, $i = 1,\dots,m$, are the uncertain parameters and each u_i belongs to the uncertainty set $\mathcal{U}_i$ described by convex quadratic inequalities:

$$\mathcal{U}_i := \left\{u_i \in \mathbb{R}^s : \frac{1}{2}u_i^T B_i^l u_i + (b_i^l)^T u_i + \beta_i^l \le 0,\ l = 1,\dots,q\right\}, \tag{1}$$

with positive semidefinite $(s \times s)$ symmetric matrices B_i^l, $b_i^l \in \mathbb{R}^s$, $\beta_i^l \in \mathbb{R}$, $i = 1,\dots,m$, $l = 1,\dots,q$. It is a robust counterpart of the polynomial optimization (POP) with constraint data uncertainty of the form

$$(POP) \quad \min_{x\in\mathbb{R}^n} \quad f(x)$$
$$\text{subject to} \quad g_i^0(x) + \sum_{j=1}^{s} u_i^j g_i^j(x) \le 0, \quad i = 1,\dots,m.$$

Note that we assume that the objective function f is free of uncertainty because the uncertainty in the objective can be converted to the form of (RP) by introducing additional variables [1].

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http://dx.doi.org/10.1016/j.orl.2017.04.005
0167-6377/

The robust problem (RP) covers a broad class of robust optimization problems because many commonly used uncertainty sets, such as ball and box uncertainty sets as well as the intersection of ellipsoidal uncertainty sets, discussed in the literature [1], can be expressed in terms of convex quadratic inequalities. A successful treatment for the special case of (RP), where f and g_i^j are linear, or convex quadratic functions, has already been given in the literature [1,2,5] with the focus on reformulating them as equivalent semidefinite programming problems (SDPs). Exact SDP relaxations are also known in the literature for robust SOS-convex programming problems under various classes of uncertainty sets [9,10].

Unfortunately, the robust optimization problem (*RP*) is, in general, a very hard problem numerically and does not admit an exact reformulation to a semidefinite linear programming problem because finding a global solution of (RP) is an NP-hard problem even in the case where the polynomials are general convex polynomials [5] and the uncertainty sets are ellipsoids. Moreover, the underlying global polynomial optimization problem without uncertainty is already known to be a hard problem to solve numerically [6,16,15]. To our best knowledge, a study of solving the general robust polynomial optimization problem (*RP*) under constraint data uncertainty does not appear to be available in the literature [3,1]. For a recent related contribution where the uncertainty occurs *only* in the objective function, see [15].

The purpose of this paper is to propose "numerically tractable" convergent SDP approximation schemes for (*RP*) by employing powerful semi-algebraic representation result, known as Putinar's theorem. We show that a convergent hierarchy of SDP relaxations holds for (RP), which, in particular, also applies to the corresponding known NP-hard problems with the intersection of ellipsoidal uncertainty. We provide numerical examples including an example with 20 variables illustrating our main result. We also discuss the scalability and implementability of our approach for larger size problems in Remark 2.1. We prove that the optimal values of the SDP relaxations to (*RP*) converge to the global optimal value of (*RP*). We also show how known exact relaxation results can be deduced from the proposed hierarchical relaxation scheme for special classes of robust convex quadratic programs.

Throughout this paper, we assume that the feasible set of (*RP*) is nonempty, and the problem (*RP*) is **regular**, that is, there exists one function $p = \mu_0 f + \sum_{i=1}^m \mu_i(g_i^0 + \sum_{j=1}^s \hat{u}_i^j g_i^j)$, for some $(\hat{u}_i^1, \ldots, \hat{u}_i^s) \in \mathcal{U}_i, \mu_i \geq 0, i = 0, 1, \ldots, m$ and $\sum_{i=0}^m \mu_i = 1$, such that p is coercive in the sense that $\lim_{\|x\|\to\infty} p(x) = +\infty$. We note that this regularity assumption not only guarantees that a global solution for (RP) exists but also can easily be numerically checked for polynomial optimization problems (see for example, [7,8]).

The outline of the paper is as follows. Section 2 defines a hierarchy of SDP relaxations for robust polynomial problems under uncertainty sets that are described by convex quadratic inequalities and establishes its convergence. Section 4 concludes the paper and presents some future research topics.

2. Convergent relaxations & intersection of ellipsoidal uncertainty

In this section, we propose a sequential SDP relaxations for (RP) and establish its convergence under suitable assumptions.

We begin this Section by recalling some basic notations for matrices. We denote S^n the space of symmetric $n \times n$ matrices with the trace inner product given by $\langle A, B\rangle = \mathrm{Tr}[AB] = \sum_{1\leq i,j\leq n} A_{ij}B_{ij}$, and $\succeq$ denotes the Löwner partial order of S^n, that is, for $M, N \in S^n, M \succeq N$ if and only if $(M - N)$ is positive semidefinite. In particular, $M \succeq 0$ means that M is positive semidefinite. Let $S^n_+ := \{M \in S^n : M \succeq 0\}$ be the closed convex cone of positive semi-definite $n \times n$ (symmetric) matrices, equipped with the trace inner product. The Frobenius norm of a matrix M is denoted by $\|M\|_F$ and is defined as $\|M\|_F = \sqrt{\sum_{1\leq i,j\leq n} M_{ij}^2}$. Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We use diag$(a)$ to denote the diagonal $(n \times n)$ matrix whose ith diagonal elements equals to $a_i, i = 1, \ldots, n$. Moreover, the $(n \times n)$ identity matrix is denoted by I_n.

We also recall some basic definitions for polynomials. We say that a real polynomial f is sums-of-squares (cf. [16]) if there exist real polynomials $f_j, j = 1, \ldots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set consisting of all sum of squares real polynomials in the variable x is denoted by $\Sigma^2[x]$. Moreover, the set consisting of all sum of squares real polynomials with degree at most d is denoted by $\Sigma_d^2[x]$. For a polynomial f, we use deg$f$ to denote the degree of f.

Recall [16, Definition 2.2] that a symmetric matrix polynomial $U \in \mathbb{R}[x]^{m\times m},\ x \in \mathbb{R}^n$, is said to be an SOS-matrix polynomial if there exists a possibly non-square matrix polynomial $W(x)$ with m columns such that $U(x) = W(x)^T W(x)$. Denote $\mathrm{M}\Sigma^2[x]$ (more explicitly, $\mathrm{M}\Sigma_m^2[x]$) the set of all symmetric polynomial in $\mathbb{R}[x]^{m\times m}$. Clearly, if $U \in \mathrm{M}\Sigma^2[x]$ then, for each $x \in \mathbb{R}^n,\ U(x)$ is a positive semidefinite symmetric matrix. Note that there exist some different notions of SOS-matrix polynomial in the literature [14].

The following matrix Putinar's theorem [19, Corollary 1] provides a representation for positivity of a real polynomial over a set described by a polynomial matrix inequality. This result will play a key role in our convergence proof of the semi-definite programming relaxation scheme later in this section.

Lemma 2.1 (*Matrix Putinar's Theorem cf. [19]*). *Let f be a real polynomial on $\mathbb{R}^n$. Let $K = \{x \in \mathbb{R}^n : G(x) \succeq 0\}$ where $G : \mathbb{R}^n \to S^q$ is a matrix polynomial. Suppose that $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ is compact for some polynomial p such that $p = \overline{\sigma}_0 + \langle \overline{R}, G\rangle$, where $\overline{\sigma}_0$ is a sum-of-squares polynomial and $\overline{R}$ is a sum-of-squares matrix polynomial. If f is positive over K, then there exist sums-of-squares polynomial σ_0 and a sums-of-squares matrix polynomial $R : \mathbb{R}^n \to S^q$ such that $f = \sigma_0 + \langle R, G\rangle$.*

Throughout this section, we use the following assumptions:

Assumption 2.1. We assume that $\mathcal{U}_i$ satisfies the Slater condition, i.e., there exists $\tilde{u}_i \in \mathcal{U}_i$ such that $\frac{1}{2}\tilde{u}_i^T B_i^l \tilde{u}_i + (b_i^l)^T \tilde{u}_i + \beta_i^l < 0$, for all $l = 1, \ldots, q,\ i = 1, \ldots, m$, and (*RP*) is regular, that is, there exists a function

$$p = \mu_0 f + \sum_{i=1}^m \mu_i \left(g_i^0 + \sum_{j=1}^s \hat{u}_i^j g_i^j \right),$$

for some $\hat{u}_i = (\hat{u}_i^1, \ldots, \hat{u}_i^s) \in \mathcal{U}_i, \mu_i \geq 0, i = 0, 1, \ldots, m$ and $\sum_{i=0}^m \mu_i = 1$ such that p is coercive in the sense that $\lim_{\|x\|\to\infty} p(x) = +\infty$.