Operations Research Letters 45 (2017) 342–347

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Drift analysis of ant colony optimization of stochastic linear pseudo-boolean functions

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ARTICLE INFO

Article history:
Received 9 April 2016
Received in revised form
4 May 2017
Accepted 4 May 2017
Available online 10 May 2017

Keywords:
Drift analysis
Runtime analysis
Ant System
Global optimization

ABSTRACT

In this paper we study the behavior of a variant of the Max–Min Ant System algorithm when applied to a stochastic Linear Pseudo-Boolean Optimization problem. Previous related work is on a partial analysis of its performance on a different problem. Here, we carry out its complete performance analysis giving a bound on its average runtime using drift analysis. For the purpose, we give a new drift theorem and use it to analyze the algorithm when applied to our problem.

1. Introduction

Drift analysis is an increasingly popular technique for the runtime analysis of randomized search algorithms [25,5,6]. Several variants based on variable drift theorems have already been proposed, for instance in [22,18,3]. Here, we set out to use the drift analysis method to analyze the behavior of an Ant Colony Optimization (ACO) algorithm when applied to the Linear Pseudo-Boolean Optimization (LPBO) problem. The choice of this algorithm is due to it being a powerful bio-inspired meta-heuristic. It was first described in [7]. ACO is inspired by the complex behavior of ant colonies which exhibit the so called swarm intelligence. This emergent intelligence results from the simple behavior of individual ants which, using pheromone as an indirect communication mechanism, confer to the colony a complex behavior on par with that of higher level organisms. It has been applied successfully to a wide range of problems arising in combinatorial as well as stochastic, dynamic and continuous optimization, [8,20]. Here we consider the Single-Destination Shortest Path (SDSP) problem. When the algorithm is not directly applicable to a given problem, the latter is transformed to SDSP.

In general, ACO algorithms are used when a solution is made up of many components; artificial ants then build solutions by successively selecting appropriate components. Pheromone is added to components that often belong to good solutions, while it evaporates from others. The amount of the pheromone update is controlled by the so-called evaporation factor ρ; when set low, it leads to a slower but wider coverage of the search space. This usually allows for better solutions to be found but at the cost of a longer runtime.

There are several papers which analyze the behavior of ACO algorithms. The first study is on proofs of convergence [11,12]. Recent advances concern the runtime of ACO algorithms when applied to combinatorial problems such as the Minimum Spanning Tree (MST) [23], TSP [19], SDSP [9] and Pseudo-Boolean functions [20].

In this paper, we are particularly interested in the work of [9] which is concerned with analyzing the performance of a new version of MMAS called MMAS-fp-norm, when applied to SDSP. According to [9], MMAS-fp-norm compares well with other variants of MMAS from an experimental point of view. But, their theoretical study focuses mainly on getting an upper bound on the expected first hitting time. This is not sufficient; a definite conclusion on the performance of MMAS-fp-norm is therefore still lacking due to the missing lower bound which allows to conclude on the bad performance of the algorithm contrary to the upper bound which only allows to conclude on its good performance. Here, we endeavor to fill this gap focusing on the study of the MMAS-fp-norm algorithm

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http://dx.doi.org/10.1016/j.orl.2017.05.002
0167-6377/

performance when applied to the stochastic version of LPBO. The latter is first converted into a stochastic SDSP problem, then we analyze and derive a lower bound on the average runtime of MMAS-fp-norm and improve the upper bound given in [9]. As will be seen later, (Section 4.4), the analysis of LPBO is a convenient way to measure the strength and weakness of a given ACO-type algorithm.

The transformation of stochastic LPBO to stochastic SDSP results in a general form of the model as given in [20]. Experiments have shown that ACO algorithms can be particularly effective on problems involving uncertainty [1]. The first results in this field are given in [13,14], where a formal analysis of ACO algorithms has been provided and its convergence to the desired solution shown. In [10] it has been shown that ACO is robust even for an arbitrarily large noise in the fitness function. In [5,17] a rigorous analysis of the expected runtime of some variants of MMAS when applied to stochastic problems, is given.

Here, we analyze a MMAS-fp-norm algorithm which is based on a fitness proportional pheromone update, when applied to the general case of the LPBO problem defined on binary strings of length n. This scheme has been used in practice [26]. In each iteration, the amount of updated pheromone always depends on the quality of the new solution, and not the best-so-far solution by contrast to other algorithms. This mechanism uses an implicit averaging to find the best solution.

For the mathematical analysis, we propose a new variant of the variable drift theorem which we prove using the general drift theorem given in [21]. It is a generalization of drift theorems which can be found in the literature, as in [22,18,4], for the upper bound and [16,3,27,2], for the lower bound. The process corresponding to the MMAS algorithm applied to LBPO on a stochastic graph takes continuous values. As the pheromone level is updated with the results of random fitness evaluations, uncountably many different pheromone values are possible in each iteration.

The paper is organized as follows. In Section 2, we present a variable drift theorem which gives a lower bound on the expected first hitting time. We define formally the considered problem and the algorithm to solve it in Section 3. In Section 4, we present our results on the runtime of MMAS-fp-norm. Section 5 is the conclusion and further work.

2. The variable drift theorem

This theorem, already mentioned above and used to establish the main results of this paper, is given here first. Before that, however, we first recall the theorem below, (part of Theorem 2 in [21]).

Assume that the stochastic process $(X_t)_{t\geq 0}$ is reduced to its natural filtration $F_t := (X_0; \dots ; X_t)$, i.e., the information available up to time t and let the first hitting time $T = \min\{t|X_t = 0\}$. Let $\Delta X_t = X_t - X_{t+1}$.

Theorem 1 (*General Drift, [21]*). *Let $(X_t)_{t\geq 0}$, be a stochastic process over a state space $S = \{0\} \cup [x_{\min}, x_{\max}]$, where $x_{\min} > 0$. Furthermore, let $h : [x_{\min}, x_{\max}] \longrightarrow \mathbb{R}^+$ be a continuous function and define $g : S \longrightarrow \mathbb{R}^{\geq 0}$ by $g(x) := \frac{x_{\min}}{h(x_{\min})} + \int_{x_{\min}}^{x} \frac{1}{h(y)} dy$ for $x \geq x_{\min}$ and $g(0) = 0$. Then for $X_t \geq x_{\min}$, if $E(\Delta X_t|F_t) < h(X_t)$ and $E(\Delta g(X_t)|F_t) < \alpha_l$ for some $\alpha_l > 0$, then*

$$E(T|X_0) > \frac{g(X_0)}{\alpha_l}.$$

This drift theorem is too complicated to apply directly. Some additional assumptions are needed. The difficulty is in finding assumptions which must not only be satisfied by the problem studied, but also allow us to find parameter-free bounds α_u and α_l. However, it can be applied with a different distance g. In some cases, a different distance is more appropriate as in [15]. Consider Theorem 2, which uses the assumption of continuity of function h.

Theorem 2. *Let $(X_t)_{t\geq 0}$, be a stochastic process over a state space $S = \{0\} \cup [x_{\min}, x_{\max}]$, where $x_{\min} > 0$. Assume that for $X_t \geq x_{\min}$ there exist two continuous functions d_1 and d_2 such that $d_1(X_t) \leq \Delta X_t \leq d_2(X_t)$ and h, h^+ and h^- be the continuous functions on $[x_{\min}, x_{\max}]$ to $\mathbb{R}^+$ such that, if $E(\Delta X_t|F_t) \leq h(X_t)$, $E\left(\Delta X_t \mathbf{1}_{\{X_t \geq X_{t+1}\}}|F_t\right) \leq h^+(X_t)$, $E\left(\Delta X_t \mathbf{1}_{\{X_t < X_{t+1}\}}|F_t\right) \leq h^-(X_t)$ and $\alpha_l > 0$ such that*

$$\alpha_l \geq \sup_{u\in[x_{\min},x_{\max}]} \left(\frac{h^+(u)}{\inf\limits_{v\in J} h(v)} + \frac{h^-(u)}{\sup\limits_{v\in J} h(v)} \right),$$

for $J = \{v \in [x_{\min}, x_{\max}] : d_1(u) \leq u - v \leq d_2(u)\}$, then we have

$$E(T|X_0) \geq \frac{g(X_0)}{\alpha_l}.$$

Proof. Since h is a continuous function then $g \in C^1([x_{\min}, x_{\max}])$. Using the mean value theorem with $g'(x) = 1/h(x)$ for all x, we get

$$\frac{y-x}{\sup\limits_{[x,y]} h(t)} \leq g(y) - g(x) \leq \frac{y-x}{\inf\limits_{[x,y]} h(t)},$$

where $x, y \in [x_{\min}, x_{\max}]$ with $x \leq y$. Thus, with $X_t > 0$ we get

$$\begin{aligned} &E(\Delta g(X_t)|F_t; X_t = u) \\ &\quad \leq \frac{h^+(u)}{\inf\limits_{y\in J} h(v)} + \frac{h^-(u)}{\sup\limits_{y\in J} h(v)} \\ &\quad \leq \sup_{u\in[x_{\min},x_{\max}]} \left(\frac{h^+(u)}{\inf\limits_{y\in J} h(v)} + \frac{h^-(u)}{\sup\limits_{y\in J} h(v)} \right) \end{aligned}$$

which gives the results after applying Theorem 1. □

Note that the above theorem can also be applied if the probability $1 - \Pr(J)$ is negligible. It has two characteristics. The first is that there are few constraints to satisfy, apart from the assumption of continuity, such as the constraint of the slowly changing process according to the search space which is necessary for its successful use in ACO-type Algorithms for instance. The second is that it gives a lower bound on the expected hitting time: there are fewer drift theorems for the lower bound than for the upper bound, (see [21]). However, these theorems are important to draw complete conclusions on the performance of a given algorithm, if any.

3. Ant colony optimization algorithms

We first introduce the stochastic LPBO problem, formally. We then present algorithm MMAS-fp-norm and explain how it can be applied to this problem before fully analyzing it in Section 4.

3.1. Problem definition

Define a linear pseudo-Boolean function f as

$$\forall x = (x_1, \dots, x_n)^T \in \{0, 1\}^n : f(x) = \sum_{i=1}^{n} w_i x_i \in \mathbb{R},$$

with weights $(w_i)_{i=1,\dots,n} \in \mathbb{R}$. We only consider positive weights since a function with negative weights w_i may be transformed into a function with positive weights $w_i' = -w_i$ by exchanging the meaning of bit values 0 and 1 for bit i. The result is a function whose value is increased by a larger additive term w_i'. These exchange