Operations Research Letters 45 (2017) 366–370

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On a duration problem with unbounded geometrical horizon

Marek Skarupski

Wrocław University of Science and Technology, Faculty of Pure and Applied Mathematics, Wyb. Wyspiańskiego 27, 50-370, Wrocław, Poland

ARTICLE INFO

Article history:
Received 27 October 2016
Received in revised form 1 May 2017
Accepted 1 May 2017
Available online 17 May 2017

Keywords:
Optimal stopping
Duration problem
Secretary problem
Non-extremal observations

ABSTRACT

We consider a full information duration problem with random, unbounded horizon. We choose the object and hold it as long as it will be relatively best or second best object. We show that the problem of stopping on two first order statistics reduces to the problem on stopping only on the relatively best object. We derive the optimal strategy and the value of the problem. The similarities between duration problem and the best or second best choice problem are revealed.

1. Introduction

Assume that at moment n we are interested in choosing relatively best or second best object. The observations come from a known distribution and we observe its exact value. In the literature is known as a class of full information stopping problems. We choose the object and hold it as long as it will be relatively best or second best object. The concept of duration of owning the relatively best object was introduced by Ferguson, Hardwick and Tamaki [3]. We observe sequentially $X_1, \dots, X_N$ i.i.d. random variables with known distribution uniform on the interval [0, 1]. The problem in this paper refers to the models from the above article as well as to the concept presented by Porosiński and Szajowski [13] in the sense that the duration is based on the 1st and 2nd order statistics. We consider a special case where N is a random variable geometrically distributed, i.e.

$$P(N = k) = p_k = pq^{k-1}, \quad 0 < p < 1; q = 1 - p, k = 1, 2, \dots. \tag{1}$$

The paper deals with generalization of the models mentioned. Following Kurushima and Ano [1], we consider the problem of stopping on the relatively best object.

1.1. Previous works

The optimal stopping problems with unbounded random horizon are extension of the problems with finite horizon. In the works by Gilbert and Mosteller [4] and Presman and Sonin [14], it is observed that the optimal strategies could have unexpected forms. Generally, the topic can be divided into two groups: no-information case and full-information case. Best choice problem for full-information case has been shown by Gilbert and Mosteller [4]. In Porosiński [10], the idea of random discrete horizon is applied. Petruccelli [9] allowed a backward solicitation in the problem. The application of random horizon and its distribution affects the reward function and the Markov chain in the stopping problem (cf. Porosiński [11]). The duration of owning the best object was presented by Ferguson, Hardwick and Tamaki [3]. The explicit formulas for the optimal payoff function were given for the duration problem by Mazalov and Tamaki [8]. Tamaki [21] solved the problem of stopping on the relatively best object with random horizon. This idea was applied to the geometrical horizon by Porosiński, Skarupski and Szajowski [12]. Gnedin [5] used the concept of geometrical horizon to find the rule for selection of an increasing subsequence from a random sample. The similarities between best choice problems and duration of owning the best objects was recognized also by Gnedin [6]. Idea of stopping on the relatively best and holding it as long as it will be best or second best object in full information case was presented by Kurushima and Ano [1]. In this work, the authors do not consider stopping on the relatively second observation and they considered a problem with a finite horizon.

A problem of stopping on the relatively best or second best object when the number of objects is random was solved by Kawai and Tamaki [7]. However, authors considered a problem where the horizon variable N is bounded with probability one and they focused on no-information case. A similar problem was considered by Tamaki [19]. The problem of stopping on the relatively best object with random horizon was solved by the same author in [20]. Although it was a great work on this topic, it does not consider the cases with random unbounded horizon. There are examples of

E-mail address: Marek.Skarupski@pwr.edu.pl.

http://dx.doi.org/10.1016/j.orl.2017.05.001
0167-6377/

distributions of the horizon, but all of them are on finite support. The most similar work to this one was done by Szajowski and Tamaki [17] where the problem for no-information case was described. Some results are also given by Tamaki [18]. An interesting problem related to the duration problem with multiple stopping was considered by Pearce Szajowski and Tamaki [22]. The problem for full information best or second best choice problem when the number of observations is finite and known remains unsolved.

1.2. The plan of the paper

The paper is organized as follows. In Section 2, we consider a duration problem where it is allowed to stop only on the best observations. The solution of this problem is presented in two ways. In Section 3, we consider a problem where it is allowed to stop on the best or second best object. We show that this problem reduces to the previous one. Moreover, we compare the strategies for two different problems: best-choice and duration.

2. Stopping on the relatively best object

Let X_n denote the value of the nth object. Let $w(n,x)$ denote the payoff function for stopping on the nth object whose value is x. Moreover, let T_n be a random variable that denotes the moment after time n when better observation occurs. The payoff is a time period of owning the object as long as its range remains within 1 and 2. The duration is calculated as long as the rank of the object will not be greater than 2 or as long as the time horizon will not terminate.

$$\begin{aligned} w(n,x) &= E[T_n - n | X_n = x] \\ &= \sum_{k=1}^{\infty} k(k-1)x^{k-2}(1-x)^2 \frac{\pi_{n+k}}{\pi_n} + \sum_{k=1}^{\infty} kx^k \frac{p_{n+k}}{\pi_n} \\ &\quad + \sum_{k=1}^{\infty} k^2 x^{k-1}(1-x)\frac{p_{n+k}}{\pi_n} \\ &= -2(x-1)^2 \frac{q^2}{(qx-1)^3} + p\frac{qx}{(qx-1)^2} \\ &\quad - (1-x)pq\frac{qx+1}{(qx-1)^3} \\ &= \frac{2q}{1-qx} - \frac{q(1-q)}{(1-qx)^2}, \end{aligned} \tag{2}$$

where $\pi_k = \sum_{j=k}^{\infty} p_j = q^{k-1}$. Let $Tw(n,x)$ denote the payoff when the decision maker does not accept the relatively best applicant whose value $X_n = x$ is the maximum value among that of the applicants arrived so far and accepts the next first relatively best applicant. Then,

$$\begin{aligned} Tw(n,x) &= \sum_{m=n+1}^{\infty} \int_x^1 w(n,v)\frac{\pi_m}{\pi_n} x^{m-n-1} dv \\ &= \sum_{m=n+1}^{\infty} \Big[\int_x^1 \frac{2q}{1-qv}\frac{(qx)^{m-n}}{x} dv \\ &\quad - \int_x^1 \frac{q(1-q)}{(1-qv)^2}\frac{(qx)^{m-n}}{x} dv \Big] \\ &= \frac{q}{1-qx}\Big(2\log\big(\frac{qx-1}{q-1}\big) + \frac{1-q}{1-qx} - 1\Big). \end{aligned} \tag{3}$$

Denote $G(n,x) := w(n,x) - Tw(n,x)$. The 1-Step-Look-Ahead (1-SLA) rule is described as a set

$$B = \{(n,x) : G(n,x) \geq 0\}. \tag{4}$$

2.1. The first proof of the 1-SLA optimality

To show that 1-SLA rule is optimal, it is necessary to show two statements: (i) $G(n,x) \geq 0 \Rightarrow G(n+k,x) \geq 0$, $k = 1,2,\ldots$ and (ii) $G(n,x) \geq 0 \Rightarrow G(n,y) \geq 0$, $y \geq x$. Since the payoff does not depend on n statement, (i) is obvious. It is enough to show the second point. Consider a function $g(x) := G(n,x)$. We show that if $g(x) \geq 0$ for some x, then the function will also be greater than 0 for $y > x$. It is equivalent to the ascertainment that if for some x holds the inequality

$$3 - 2\frac{1-q}{1-qx} - \log\big(\frac{1-qx}{1-q}\big)^2 \geq 0, \tag{5}$$

then it holds for all $y > x$. Calculate the derivative with respect to x of the LHS of (5). It is equal to

$$-2q(1-q)\frac{1}{(1-qx)^2} + 2q(1-qx)\frac{1}{(1-qx)^2} = \frac{2q^2(1-x)}{(1-qx)^2}$$

and it is non-negative for all $x < 1$. In point $x = 1$, the LHS of (5) reaches the value 1. So, if the inequality holds for some x, then it is true for $x < y < 1$.

We get that 1-SLA rule is optimal. Because it is the threshold rule, we get that the threshold is a value

$$x_0 = \frac{x^\star p - 1}{p - 1}, \tag{6}$$

where $x^\star$ is a solution of the equation

$$x^2 e^{\frac{2}{x}} = e^3, x > 1.$$

Numerically, $x^\star \approx 3.3145$. Note that if $p \geq \frac{1}{x^\star} := p^\star \approx 0.3017046$ the threshold is equal to 0. Therefore, 1-SLA calls for a stop on a very first observation. It is easy to see that $p^\star$ is a solution of the equation

$$p^2 e^{-2p} = e^{-3}$$

for p between 0 and 1.

2.2. The second proof of the 1-SLA optimality

Another possibility to prove the 1-SLA optimality is by using the Bellman optimality equation. Let $c_k(x)$ denote the payoff earned by continuing observations in an optimal manner. Then, $v_k(x) = \max\{w_k(x), c_k(x)\}$ is the optimal payoff provided that we start from state (k,x). It leads to the recursive equation:

$$c_k(x) = \frac{\pi_{k+1}}{\pi_k}\left(x c_{k+1}(x) + \int_x^1 v_{k+1}(y) \right). \tag{7}$$

Since w does not depend on k also c and v do not depend on k. Therefore, Eq. (7) leads to

$$c(x) = q\left(x c(x) + \int_x^1 v(y)dy \right) \tag{8}$$

and $v(x) = \max\{w(x), c(x)\}$. We get

$$c(x) = \frac{q}{1-qx}\int_x^1 v(y)dy$$

and the optimal payoff $v(x) = \frac{q}{1-qx}\{2 - \frac{1-q}{1-qx}, \int_x^1 v(y)dy\}$. For big values of x, let us say $x > x_0$; we have $v(x) = w(x)$, where x_0 satisfies an equation

$$2 - \frac{1-q}{1-qx} = \int_x^1 v(y)dy.$$

It means that the stopping region contains $\{(n,x) : x \geq x_0\}$. For $x \leq x_0$, we have

$$(1-qx)v(x) = q\left(\int_x^{x_0} v(y)dy + \int_{x_0}^1 w(y)dy \right) \tag{9}$$