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Price competition under linear demand and finite inventories: Contraction and approximate equilibria

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ABSTRACT

We consider a multi-period price competition among multiple firms with limited inventories of substitutable products, and study two types of equilibrium: with and without recourse. Under a linear demand model, we show that an equilibrium without recourse uniquely exists. In contrast, we show an equilibrium with recourse need not exist, nor be unique. In a low-influence regime, using the equilibrium without recourse, we construct an approximate equilibrium with recourse with the same equilibrium price trajectory.

1. Introduction

In many practical situations, multiple firms selling substitutable products set their prices competitively to sell limited inventories over a finite selling horizon, given that the demand of each firm jointly depends on the prices charged by all firms. For example, airlines competitively set the prices for their limited seat inventories in a particular market. Firms selling electronic products take the prices of their competitors into consideration when setting their prices. In this paper, we consider multiple firms with limited inventories of substitutable products. Each firm chooses the prices that it charges for its product over a finite selling horizon. The demand that each firm faces is a deterministic function of the prices charged by all of the firms, where the demand of a firm is linearly decreasing in its price and linearly increasing in the prices of the other firms. Each firm chooses its prices over a finite selling horizon to maximize its total revenue.

Main Contributions. We study two types of equilibrium for the competitive pricing setting described above. In an equilibrium without recourse, at the beginning of the selling horizon, each firm selects and commits to the prices it charges over the whole selling horizon, assuming that the other firms do the same. In an equilibrium with recourse, at each time period in the selling horizon, each firm observes the inventories of all of the firms and chooses its price at the current time period, again under the assumption that the other firms do the same. Essentially, an equilibrium without recourse corresponds to an open-loop equilibrium [4], whereas an equilibrium with recourse corresponds to a Markov perfect equilibrium (MPE) [5] in the dynamic game among the firms. Despite the fact that the demand of each firm is a deterministic function of the prices so that there is no uncertainty in the firms' responses, we show a clear contrast between the two equilibrium notions.

We consider the diagonal dominant regime, where the price charged by each firm affects its demand more than the prices charged by the other firms. In other words, if all of the competitors of a firm decrease their prices by a certain amount, then the firm can decrease its price by the same amount to ensure that its demand does not decrease. This regime is rather standard in the existing literature and it is used in, for example, [2] and [6]. Focusing on the equilibrium without recourse, we show in Section 2 that the best response of each firm to the price trajectories of the other firms is a contraction mapping, when viewed as a function of the prices of the other firms. In this case, it immediately follows that the equilibrium without recourse always exists and it is unique (see [17, Section 2.5]).

We give counterexamples in Section 3 to show that an equilibrium with recourse may not exist or may not be unique. Motivated by this observation, we look for an approximate equilibrium that is guaranteed to exist. We call a strategy profile for the firms an ϵ-equilibrium with recourse if no firm can improve its total revenue by more than ϵ by deviating from its strategy profile. We consider a low influence regime, where the effect of the price of a firm on the demand of another firm is diminishing, which naturally holds when the number of firms is large. We show in Section 4

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that the equilibrium without recourse can be used to construct an ϵ-equilibrium with recourse that has the same price trajectory as the equilibrium without recourse. So, intuitively speaking, an ϵ-equilibrium with recourse is expected to exist when the number of firms is large.

Our results fill a gap in a fundamental class of revenue management problems. Although there is no uncertainty in the firms' responses, the equilibria with and without recourse are not the same concept and can be qualitatively quite different. While the equilibrium without recourse uniquely exists, the same need not hold for the equilibrium with recourse. Also, our contraction argument for showing the existence and uniqueness of the equilibrium without recourse uses the Karush–Kuhn– Tucker (KKT) conditions for the firm's problem. Though contraction arguments are standard for showing existence and uniqueness of equilibrium [17], to the best of our knowledge, this duality-based contraction argument is new for price competition under limited inventories. This argument becomes surprisingly effective when dealing with linear demand functions, but it is an open question whether similar arguments hold for other demand functions. Lastly, our results indicate that in a low influence regime the equilibrium without recourse can be used to construct an ϵ-equilibrium with recourse with the same price trajectory as the equilibrium without recourse.

Literature Review. Similar to us, [6] considers price competition among multiple firms with limited inventories over a finite selling horizon. There are three key differences between their work and ours. First, they focus on a continuous-time setting, whereas we study a discrete-time formulation. Second, they consider a generalized Nash game [16] where each firm considers all firms' capacity constraints while setting their prices, whereas in our model, each firm only considers its own capacity constraints. Most importantly, they focus on open-loop and closed-loop equilibria, and show that in the diagonally dominant regime a unique open-loop equilibrium exists and coincides with a closed-loop equilibrium. Although an equilibrium without recourse in our setting is the same as an open-loop equilibrium, our equilibrium with recourse is more restrictive than their closed-loop equilibrium. In particular, their closed-loop equilibria need not be *perfect*, whereas our equilibrium with recourse is a Markov perfect equilibrium. Thus, we show that the equilibrium with recourse can be different from the equilibrium without recourse. More precisely, although the former equilibrium need not exist or be unique, the latter is an approximate equilibrium with recourse in the low influence regime.

There are a number of papers that study price competition over a single period. [12] shows that pure Nash equilibrium (NE) exists for a wide class of supermodular demand models. [7] provides sufficient conditions for uniqueness of equilibrium in the Bertrand game when the demands of the firms are nonlinear functions of the prices, there is a non-linear cost associated with satisfying a certain volume of demand and each firm seeks to maximize its expected profit. [14] identifies the conditions for existence and uniqueness of pure NE when the demands are characterized by a mixture of multinomial logit models and the cost of satisfying a certain volume of demand is linear in the demand volume. [8] considers price competition among multiple firms when the relationship between demand and price is characterized by the nested logit model and provides conditions to ensure the existence and uniqueness of the equilibrium. [13] proves the existence of pure strategy equilibrium in a price competition between two suppliers when capacity is private information.

Considering the papers on price competition over multiple time periods, [9] studies a stochastic game when there are strategic consumers choosing the time to purchase. [10] studies a competitive pricing problem when the relationship between demand and price is captured by the multinomial logit model and inventory levels are public information. [1] studies the pricing game between two firms with limited inventories facing stochastic demand. The authors characterize the unique subgame perfect Nash equilibrium. [11] shows the existence of a unique pure MPE in a pricing game between two firms offering vertically differentiated products.

2. Equilibrium without recourse

There are n firms indexed by $N = \{1, \dots, n\}$. Firm i has c_i units of initial inventory, which cannot be replenished over the selling horizon. There are τ time periods in the selling horizon indexed by $T = \{1, \dots, \tau\}$. We use p_i^t to denote the price charged by firm i at time period t. Using $\boldsymbol{p}^t = (p_1^t, \dots, p_n^t)$ to denote the prices charged by all of the firms at time period t, the demand faced by firm i at time period t is given by $D_i^t(\boldsymbol{p}^t) = \alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t$, where $\alpha_i^t > 0$, $\beta_i^t > 0$ and $\gamma_{i,j}^t > 0$. We assume that the price charged by each firm affects its demand more than the prices charged by the other firms, in the sense that $\sum_{j\neq i}\gamma_{i,j}^t < \beta_i^t$ for all $i \in N$, $t \in T$. Also, using $\boldsymbol{p}_{-i}^t = (p_1^t, \dots, p_{i-1}^t, p_{i+1}^t, \dots, p_n^t)$ to denote the prices charged by firms other than firm i at time period t, to avoid negative demand quantities, we restrict the strategy space of the firms such that each firm i charges the price p_i^t at time period t that satisfies $\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t \geq 0$, given the prices $\boldsymbol{p}_{-i}^t$ charged by the other firms. If the firms other than firm i commit to the price trajectories $\boldsymbol{p}_{-i} = \{\boldsymbol{p}_{-i}^t : t \in T\}$, then we can obtain the best response of firm i by solving the problem

$$\max\Big\{ \sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) p_i^t : \sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) \leq c_i, \alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t \geq 0 \;\; \forall\, t \in T,\;\; p_i^t \geq 0 \;\; \forall\, t\in T \Big\}. \quad (1)$$

Since $\beta_i^t > 0$, problem (1) has a strictly concave objective function and linear constraints, which implies that the best response of firm i is unique.

Using the non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ for the first and the second constraint in problem (1), the KKT conditions for this problem are

$$\Big(\sum_{t\in T}\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) - c_i\Big) v_i = 0,$$
$$\Big(\alpha_i^t - \beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t\Big) u_i^t = 0 \;\; \forall\, t \in T, \quad (2)$$
$$\alpha_i^t - 2\,\beta_i^t p_i^t + \sum_{j\neq i}\gamma_{i,j}^t p_j^t + \beta_i^t (v_i - u_i^t) = 0 \;\; \forall\, t \in T.$$

Since problem (1) has a concave objective function and linear constraints, the KKT conditions above are necessary and sufficient at optimality; see [3]. In other words, for a feasible solution $\{p_i^t : t \in T\}$ to problem (1), there exist corresponding non-negative dual multipliers v_i and $\{u_i^t : t \in T\}$ that satisfy the KKT conditions in (2) if and only if $\{p_i^t : t \in T\}$ is the optimal solution to problem (1). Note that we do not associate dual multipliers with the constraints $p_i^t \geq 0$ for all $t \in T$ in problem (1) since it is never optimal for firm i to charge a negative price. Therefore, we can actually view the constraints $p_i^t \geq 0$ for all $t \in T$ as redundant constraints. We use the KKT conditions in (2) extensively to characterize the best response of firm i to the price trajectories $\boldsymbol{p}_{-i}$ of the other firms. In the rest of this section, we exclusively focus on the *strategies without recourse*, where each firm i commits to a price trajectory $\{p_i^t : t \in T\}$ at the beginning of the selling horizon and does not adjust these prices during the course of the selling horizon. If the price trajectory $\{p_i^t : t \in T\}$ chosen by each firm i is