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New counterteror measures: Feedback Nash equilibrium of a multiplayer differential game

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ABSTRACT

In this study, we consider the new trend of counterterror measures and the economic growth differential game. The feedback Nash equilibrium of this game is acquired by dynamical programming. We also characterize the feedback strategies for the government and terrorist organization. The optimal strategy in our study is then compared with a two-player zero-sum feedback equilibrium reported previously. Finally, we discuss the effectiveness of the counterterror game.

1. Introduction

A new global viewpoint regarding counterterrorism emerges that the method for effective suppression of terrorist organizations is to cultivate new forces that can compete with them. The local new forces cultivated in this way are familiar with the terrorist organizations and also superior with good timing, geographical convenience and good human relations. In this study, we will build models of counterterror games under this new trend.

Recently, as terrorist attacks become increasingly rampant, "terrorism" has received extensive attention from various research fields. Great efforts in social investigation, statistical analysis and mathematical proof have been devoted to finding out resolutions to terrorism. Owing to the intense rivalry between governments and terrorist organizations, the theory of optimal control and the game theory are commonly used to build relevant models ([2–7]). A differential game equilibrium considering both counterterror measures and economic growth was first proposed [9]. In this differential game, the open-loop Nash equilibrium was investigated, but this solution was only related to time and the initial status [9]. Later, the feedback saddle point to this issue, or namely the equilibrium solution related to the current status, was determined [8]. Both the above studies focused on zero-sum differential games, but in the present study, we will discuss a multiplayer nonzero-sum game and analyze its feedback equilibrium solution.

2. Model and definition

We inherit some marks and definitions from Refs. [9]–[8]. Then we build the counterterror measures and economic growth differential game under the new trend and determine the feedback Nash equilibrium.

The differential game under the new trend involves three players: a terrorist organization (player 1), the government (player 2), and the new force cultivated by the government (player 3). The resources that the three players have are state variables $x(t)$, $y(t)$ and $z(t)$, respectively. Then the dynamic system is expressed as follows:

$$\begin{cases} \dot{x}(t) = r_1 x(t) - (u_1(t)y(t))^a (v(t))^s (w(t))^b, & x(t_0) = x_0 > 0, \\ \dot{y}(t) = \alpha(1 - u_1(t) - u_2(t))y(t) - f v(t), & y(t_0) = y_0 > 0, \\ \dot{z}(t) = r_2[z(t) + (u_2(t)y(t))^m] - (w(t))^b, & z(t_0) = z_0 > 0. \end{cases} \tag{1}$$

The target functions that the three players expect to minimize are:

$$J_1(v, u_1, u_2, w) = \int_{t_0}^{\infty} \mathrm{e}^{-\rho(t-t_0)}[ly(t) - cx(t) - kv(t)]\mathrm{d}t, \tag{2}$$

$$J_2(v, u_1, u_2, w) = \int_{t_0}^{\infty} \mathrm{e}^{-\rho(t-t_0)}[-ly(t) + cx(t) + kv(t) - qz(t)]\mathrm{d}t, \tag{3}$$

$$J_3(v, u_1, u_2, w) = \int_{t_0}^{\infty} \mathrm{e}^{-\rho(t-t_0)}[cx(t) - qz(t)]\mathrm{d}t \tag{4}$$

where $v(t) \geqslant 0$ is the terrorist attack intensity and control function of player 1 ; $u_1(t) \geqslant 0$ and $u_2(t) \geqslant 0$ are the control functions of player 2, which are the proportions of government's counterterror investment and new force cultivation, respectively; $1-u_1(t)-u_2(t)$ is the proportion of economic investment; $w(t) \geqslant 0$ is the counterterror intensity and control function of player 3; parameters

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$0 < a < 1, s > 1, 0 < b < 1, 0 < m < 1$,and $r_1, r_2, \alpha, f, l, c, k, q, \rho$ are all positive constants; ρ is the discount rate. We assume $\rho > r_1, \rho > r_2, \rho > \alpha$.

We give the definition of the feedback Nash equilibrium of infinite horizon differential games ([1,10]). Now consider the game

$$\max_{u_i} \int_{t_0}^{\infty} e^{-\rho(t-t_0)} g^i[x(t), u_1(t), u_2(t), \ldots, u_n(t)]\mathrm{d}t, \quad \text{for } i \in N, \tag{5}$$

subject to the dynamics

$$\dot{x}(t) = h[x(t), u_1(t), u_2(t), \ldots, u_n(t)], \qquad x(t_0) = x_0. \tag{6}$$

Definition 1. For the differential game (5)–(6), an n-tuple of strategies

$\{u_i^*(t) = \phi_i^*(x) \in U_i, \text{ for } i \in N\}$

constitutes a feedback Nash equilibrium solution if there exist functionals $V^i(\theta, x)$ defined on $[0, +\infty) \times R^n$ and satisfying the following relations for each $i \in N$:

$$\begin{aligned} V^i(\theta, x) &= \int_{\theta}^{+\infty} e^{-\rho(t-t_0)} g^i[x^*(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_n^*(\eta_t)]\mathrm{d}t \\ &\geqslant \int_{\theta}^{+\infty} e^{-\rho(t-t_0)} g^i[x^{[i]}(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_{i-1}^*(\eta_t), \\ &\qquad \phi_i(\eta_t), \phi_{i+1}^*(\eta_t), \ldots, \phi_n^*(\eta_t)]\mathrm{d}t, \qquad \forall \phi_i \in \Gamma^i, \ x \in R^n, \end{aligned}$$

where in the interval $[0, +\infty)$,

$$\begin{aligned} \dot{x}^{[i]}(t) &= h[x^{[i]}(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_{i-1}^*(\eta_t), \\ &\qquad \phi_i(\eta_t), \phi_{i+1}^*(\eta_t), \ldots, \phi_n^*(\eta_t)], \qquad x^{[i]}(\theta) = x; \\ \dot{x}^*(t) &= h[x^*(t), \phi_1^*(\eta_t), \phi_2^*(\eta_t), \ldots, \phi_n^*(\eta_t)], \qquad x^*(\theta) = x, \end{aligned}$$

and η_t stands for the data set $\{x(t), x_0\}$.

3. Feedback Nash equilibrium of a multiplayer differential game

Before resolving the Hamilton–Jacobi–Isaacs (HJI) equation of this game, we simplified the value functions [8]. If

$$V(\theta, x) = -e^{-\rho(\theta-t_0)} W(x), \quad \text{for} \quad x(\theta) = x = x_\theta^* = x^*(\theta)$$

where $W(x)$ only depends on the current state x, then

$$\frac{\partial V(\theta, x)}{\partial \theta} = \rho e^{-\rho(\theta-t_0)} W(x).$$

Now we discuss the Nash equilibrium of the multiplayer nonzero-sum differential game. We introduce Proposition 2.

Proposition 2. *The feedback Nash equilibrium of the differential game (1)–(4) is given by*

$$(v^*(t), u_1^*(t), u_2^*(t), w^*(t)) = (\phi_1^*(x, y, z), \phi_2^*(x, y, z), \phi_3^*(x, y, z), \phi_4^*(x, y, z))$$

and

$$\phi_1^*(x_\theta^*, y_\theta^*, z_\theta^*) = \left\{ \frac{C}{A} \left[\frac{\alpha s B}{a(k+fB)} \right]^a \right\}^{\frac{1}{a+s}}, \tag{7}$$

$$\phi_2^*(x_\theta^*, y_\theta^*, z_\theta^*) = \frac{a(k+fB)}{\alpha s B y_\theta^*} \left\{ \frac{C}{A} \left[\frac{\alpha s B}{a(k+fB)} \right]^a \right\}^{\frac{1}{a+s}}, \tag{8}$$

$$\phi_3^*(x_\theta^*, y_\theta^*, z_\theta^*) = \left(-\frac{\alpha B}{r_2 m C} \right)^{\frac{1}{m-1}} \cdot \frac{1}{y_\theta^*}, \tag{9}$$

$$\phi_4^*(x_\theta^*, y_\theta^*, z_\theta^*) = \left(-\frac{k+fB}{sC} \right)^{\frac{1}{b}} \left\{ \frac{C}{A} \left[\frac{\alpha s B}{a(k+fB)} \right]^a \right\}^{\frac{1}{b(a+s)}}. \tag{10}$$

Moreover

$$y_\theta^* = y^*(\theta) = e^{\alpha\theta} \left(y_0 - \frac{g}{\alpha} \right) + \frac{g}{\alpha},$$

where

$$A = -\frac{c}{\rho - r_1}, \qquad B = \frac{l}{\rho - \alpha}, \qquad C = -\frac{q}{\rho - r_2},$$

$$g = \phi_1^* \left[\frac{ak + (a+s)fB}{sB} \right] + \alpha \left(-\frac{\alpha B}{r_2 m C} \right)^{\frac{1}{m-1}}.$$

Here, we will resolve a nonzero-sum differential game using the principle of dynamic programming.

Proof. Let

$$V^i(\theta, x, y, z) = -e^{-\rho(\theta-t_0)} W^i(x, y, z), \qquad i = 1, 2, 3$$

and

$$\begin{aligned} h_1 &= (u_1 y)^a v^s w^b, & h_1^* &= (\phi_2^* y)^a (\phi_1^*)^s (\phi_4^*)^b \\ h_2 &= w^b, & h_2^* &= (\phi_4^*)^b \\ h_3 &= (u_2 y)^m, & h_3^* &= (\phi_3^* y)^m. \end{aligned}$$

Then

$$\begin{aligned} \rho W^1(x, y, z) &= \min_v \left\{ ly - cx - kv + \frac{\partial W^1(x, y, z)}{\partial x}(r_1 x - h_1) \right. \\ &\quad + \frac{\partial W^1(x, y, z)}{\partial y}[\alpha(1 - u_1 - u_2)y - fv] \\ &\quad \left. + \frac{\partial W^1(x, y, z)}{\partial z}[r_2(z + h_3) - h_2] \right\} \\ &= ly - cx - k\phi_1^* + \frac{\partial W^1(x, y, z)}{\partial x}(r_1 x - h_1^*) \\ &\quad + \frac{\partial W^1(x, y, z)}{\partial y}[\alpha(1 - \phi_2^* - \phi_3^*)y - f\phi_1^*] \\ &\quad + \frac{\partial W^1(x, y, z)}{\partial z}[r_2(z + h_3^*) - h_2^*]. \end{aligned} \tag{11}$$

$$\begin{aligned} \rho W^2(x, y, z) &= \min_{u_1, u_2} \left\{ -ly + cx + kv - qz \right. \\ &\quad + \frac{\partial W^2(x, y, z)}{\partial x}(r_1 x - h_1) \\ &\quad + \frac{\partial W^2(x, y, z)}{\partial y}[\alpha(1 - u_1 - u_2)y - fv] \\ &\quad \left. + \frac{\partial W^2(x, y, z)}{\partial z}[r_2(z + h_3) - h_2] \right\} \\ &= -ly + cx + k\phi_1^* - qz + \frac{\partial W^2(x, y, z)}{\partial x}(r_1 x - h_1^*) \\ &\quad + \frac{\partial W^2(x, y, z)}{\partial y}[\alpha(1 - \phi_2^* - \phi_3^*)y - f\phi_1^*] \\ &\quad + \frac{\partial W^2(x, y, z)}{\partial z}[r_2(z + h_3^*) - h_2^*]. \end{aligned} \tag{12}$$

$$\begin{aligned} \rho W^3(x, y, z) &= \min_w \left\{ cx - qz + \frac{\partial W^3(x, y, z)}{\partial x}(r_1 x - h_1) \right. \\ &\quad + \frac{\partial W^3(x, y, z)}{\partial y}[\alpha(1 - u_1 - u_2)y - fv] \\ &\quad \left. + \frac{\partial W^3(x, y, z)}{\partial z}[r_2(z + h_3) - h_2] \right\} \\ &= cx - qz + \frac{\partial W^3(x, y, z)}{\partial x}(r_1 x - h_1^*) \end{aligned}$$