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# Error bounds for affine variational inequalities with second-order cone constraints

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ABSTRACT

In this paper, error bounds for affine variational inequalities with second-order cone constraints are considered. Examples are given to show that, in general, Lipschitz error bounds may be invalid for affine second-order cone inclusion problems. We provide a sufficient condition (not stronger than Mangasarian–Fromovitz constraint qualification), under which a local Lipschitz error bound is valid for the variational inequality problem. Moreover, under a full row rank assumption, a local Hölder error bound is established for the variational inequality problem and the Hölder exponent is bounded by a function of problem dimensions.



## 1. Introduction

Error bound conditions play an important role in the convergence analysis of numerical algorithms for optimization problems and variational inequalities. For example, the convergence analysis of matrix splitting methods, can be established under an error bound condition, for symmetric and monotone affine variational inequalities [19]. Another example is the convergence analysis of the modified Levenberg–Marquardt methods. The quadratic convergence rate can be obtained under the local error bound condition, see [33] and [4].

Error bounds for nonlinear systems, such as systems of convex functions, polynomial functions and lower semi-continuous functions, have been investigated extensively and readers are referred to [8,10–12,14,16–18,24,28–30] and the references therein.

There are many important results on error bounds for variational inequalities (VIs) over convex polyhedral sets. Motivated by the convergence analysis of matrix splitting algorithms, Luo and Tseng (1992) [19] established a local Lipschitz-type error bound for affine VIs using projection-type residual functions. The conclusion can also be obtained from error bounds for piecewise quadratic systems (see [16]). Pang (1987) [23] proved that global Lipschitz error bound holds for VIs with polyhedral constraints under the condition of strong monotonicity and Lipschitz continuity. Xiu and Zhang (2002) [31] extended these results to general VIs with convex constraints. An error bound for a system or an optimization problem is usually described in terms of residual functions. For variational inequality problems, gap functions are served as residual functions. For instance, under strong monotonicity and Lipschitz continuity, Lipschitz error bounds in terms of gap functions of VI problems can be established, see [5,25–27,32]. Moreover, these error bound results are extended to nonsmooth VIs and subanalytic VIs, see [20] and [15], respectively.

In this paper, we mainly consider error bounds for the affine variational inequalities (AVIs) with second-order cone (SOC) constraints. This variational inequality problem can be stated as: find an $x^* \in X$ satisfying

$$\langle x - x^*, \, Mx^* + q \rangle \geq 0, \; \forall x \in X, \tag{1}$$

where $X = \{x \,|\, Ax - b \in Q\}$ and $Q = \prod_{i=1}^{p} Q_{m_i+1}$ and $Q_{m_i+1}$ is a second-order cone in $\mathbb{R}^{m_i+1}$ as $i = 1, 2, \ldots, p$. A second-order cone LCP problem is a special case of Problem (1), which has found many applications in optimizations. For example, the system of KKT conditions for a quadratic programming problem with second-order cone constraints can be formulated as a second-order cone LCP problem. Lipschitz error bound condition has already been used in the convergence analysis of algorithms for solving Problem (1). For example, the convergence results for the matrix-splitting method by [6] and the proximal gradient descent method by [22] are established under the Lipschitz error bound. Since affine variational inequalities are not necessarily strictly monotone or strongly monotone, the known results are not applicable to get a Lipschitz error bound for Problem (1). To overcome this difficulty, in this paper, we propose a sufficient condition, under which a local Lipschitz-type error bound holds for Problem (1). Moreover, based

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on fractional error bounds for polynomial systems proposed by Li et. al. (2015,2016) [13,14], we establish a local Hölder error bound for Problem (1) under a full row rank condition.

The rest of this paper will be organized as follows. In the next section, examples are presented for illustrating that in general, Lipschitz error bounds may be invalid for second-order cone inclusion problems or semidefinite inequalities. Hence it also may be invalid for VIs with SOC constraints. In Section 3, we provide a sufficient condition, under which Problem (1) has a local Lipschitz-type error bound with projection-type residual functions. A local Hölder error bound for AVIs with SOC constraints is established in Section 4. It is proved that the Hölder exponent can be bounded by a function of problem dimensions. The final section makes a conclusion of the paper.

## 2. Two examples for second-order cone and semidefinite inequalities

It has been proved by Luo and Tseng (1992) [19] and Li [16] that affine variational inequalities have Lipschitz error bounds without any regularity assumptions. But for second-order cone inclusion problems and semidefinite inequalities, Lipschitz error bounds are only satisfied under some regularity conditions, see [7,9,11,21]. The following example shows that Lipschitz error bounds may be invalid for SOC problems when these regularity conditions fail.

**Example 2.1.** Consider a second-order cone inclusion problem as follows: $A_i x - b_i \in Q_i, \quad i = 1, 2,$ where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$b_1 = (1/2,\ 1,\ 0)^T, \quad b_2 = (0,\ 3/4,\ 0)^T.$$

Since $(y_0; \bar{y}) \in Q_i$ is equivalent to $\|\bar{y}\| - y_0 \le 0$, by defining

$$f_1(x_1, x_2) := \sqrt{x_1^2 + x_2^2 - 2x_2 + 1} - x_2 + \frac{1}{2}$$

$$f_2(x_1, x_2) := \left| x_2 - \frac{3}{4} \right|$$

the solution set can be formulated as

$$X = \{(x_1, x_2) \mid f_1(x_1, x_2) \le 0,\ f_2(x_1, x_2) \le 0\}.$$

It can be easily verified that $X = \{(0, \frac{3}{4})\}$. We choose the commonly used residual function

$$r(x) = \max\left\{ \left[ \sqrt{x_1^2 + x_2^2 - 2x_2 + 1} - x_2 + \frac{1}{2} \right]_+,\ \left| x_2 - \frac{3}{4} \right| \right\}.$$

Consider a sequence $\{x^k\} \subset \mathbb{R}^2 \backslash X$ defined by $x^k = (\varepsilon_k, \varepsilon_k^2 - \varepsilon_k^3 + \frac{3}{4})^T$, where $\varepsilon_k \downarrow 0$ as $k \to \infty$. We have

$$\begin{aligned}
&\lim_{k\to\infty} \frac{r(x^k)}{\mathbf{d}(x^k, X)} \\
&= \lim_{k\to\infty} \frac{\max\left\{ \frac{\varepsilon_k^2 - (\varepsilon_k^2 - \varepsilon_k^3)}{\sqrt{\varepsilon_k^2 + (\varepsilon_k^2 - \varepsilon_k^3 - 1/4)^2 + (\varepsilon_k^2 - \varepsilon_k^3 + 1/4)}},\ |\varepsilon_k^2 - \varepsilon_k^3| \right\}}{\|(\varepsilon_k, \varepsilon_k^2 - \varepsilon_k^3 + 3/4) - (0, \frac{3}{4})\|} \\
&= \lim_{k\to\infty} \frac{\varepsilon_k^2 - \varepsilon_k^3}{\sqrt{\varepsilon_k^2 + \varepsilon_k^4 - 2\varepsilon_k^5 + \varepsilon_k^6}} = 0.
\end{aligned}$$

This implies that Lipschitz error bounds do not hold for this problem.

**Example 2.2.** The second-order cone problem in Example 2.1 can be equivalently written as a semidefinite matrix inequality as follows,

$$G = \begin{pmatrix} G_1(x_1, x_2) & 0 \\ 0 & G_2(x_1, x_2) \end{pmatrix} \succeq 0, \text{ where}$$
$$G_1(x_1, x_2) = \begin{pmatrix} x_2 - 1/2 & x_1 & x_2 - 1 \\ x_1 & x_2 - 1/2 & 0 \\ x_2 - 1 & 0 & x_2 - 1/2 \end{pmatrix},$$
$$G_2(x_1, x_2) = \begin{pmatrix} 0 & x_2 - 3/4 & 0 \\ x_2 - 3/4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, semidefinite inequalities also have no Lipschitz error bounds in some cases. Note that in this example $\lambda_{min}(G_2(x)) \le 0$ for any $x \in \mathbb{R}^n$, hence the conclusion given by Deng and Hui [3] cannot be applied here to obtain a Lipschitz type error bound.

## 3. A local Lipschitz error bound for AVIs with SOC constraints

### 3.1. Preliminary

It is well known (from the convexity of $X$) that the solution set of Problem (1) is equivalent to

$$X^* = \{x \in X \mid x = [x - Mx - q]_X\}, \tag{2}$$

where $[D]_X$ denotes the orthogonal projection of $D$ onto $X$. Denote the projection $[x - Mx - q]_X$ by $z$. Then $z$ is the optimal solution of the following second-order cone constrained quadratic program,

$$\begin{cases} \min & \frac{1}{2}\|z - x + Mx + q\|^2 \\ \text{s. t.} & Az - b \in Q. \end{cases} \tag{3}$$

The KKT conditions of the problem can be written as follows.

**KKT conditions:**

$$x - z - Mx - q + A^T\lambda = 0, \tag{4}$$

$$A_i z - b_i \in Q_{m_i+1}, \quad \lambda_i \in Q_{m_i+1}, \tag{5}$$

$$\lambda_i \circ (A_i z - b_i) = 0, \quad i = 1, \ldots, p. \tag{6}$$

### 3.2. A local Lipschitz error bound

In view of Example 2.1, we have that constraint qualifications are needed for Lipschitz error bounds for Problem (1). We show in the following that Problem (1) can be reformulated as a quadratic system with linear constraints. Based on the error bound results on Lipschitz functions, we establish a local Lipschitz error bound for Problem (1) under a condition not stronger than Mangasarian–Fromovitz constraint qualification (MFCQ). For this analysis, we need the following results, which is Theorem 3.3 in [29].

**Theorem 3.1.** *Let $C \subset \mathbb{R}^n$ be a closed set, $f_i, g_j : \mathbb{R}^n \to (-\infty, +\infty]$ be l.s.c. functions for $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, s$. Assume that $x_0 \in S := \{x \in C \mid f_1(x) \le 0, \ldots, f_r(x) \le 0;\ g_1(x) = 0, \ldots, g_s(x) = 0\}$ and denote*

$$f(x) = max\{f_1(x), \ldots, f_r(x);\ |g_1(x)|, \ldots, |g_s(x)|\}.$$

*Suppose that there exist $\mu > 0$ and $\varepsilon > 0$ such that*

$$\|\xi\| > \mu^{-1}, \quad \forall \xi \in \partial_w[f(x) + \sigma_C(x)]$$

*for any $x \in C$ with $0 < f(x) < \varepsilon$ (or $\|x - x_0\| < \varepsilon$ and $0 < f(x) < +\infty$). Then we have*

$$\mathbf{d}(x, S) \le \mu[f(x)]_+ \le \mu(\|[F(x)]_+\| + \|G(x)\|),$$

*where $\partial_w f$ is $\partial_w$-subdifferential defined in [29] for l.s.c. functions and $\sigma_C(\cdot)$ is the indicator function of $C$.*