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Strategic behavior in the partially observable Markovian queues with partial breakdowns

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ABSTRACT

This paper studies the equilibrium strategies in the almost observable and almost unobservable $M/M/1$ queues with partial breakdowns. This work compensates the game theoretic analysis in Li et al. (2013) by studying the corresponding partially observable cases.

1. Introduction

Recently, there is an emerging tendency to study queueing systems from an economic viewpoint, see Burnetas and Economou [2], Guo and Hassin [3] and Yu et al. [6]. The analysis of queueing systems with breakdowns has received considerable attention in the literature. Our contribution is motivated by the paper Li et al. [4], the server's state is modulated by an external environment. Yang et al. [5] studied the *MAP*/*PH*/*N* retrial queue in a random environment. We provide supplement to Li et al. [4] by studying the corresponding partially observable cases.

2. Model description

Consider the single-server queue subject to a Poisson arrival process with rate λ. The server alternates between two states that are exponentially distributed at rates ζ and θ respectively. During the normal working state, service times are exponentially distributed with rate μ. In the partial breakdowns state, the service rate is μ_0 and $\mu_0 < \mu$. Let $L(t)$ and $I(t)$ denote the queue length and the server's state (1: normal working state, 0: partial breakdowns state) respectively. The non-zero transition rates of the process $\{(L(t), I(t)) : t \geq 0\}$ are

$$q_{(n,i)(n+1,i)} = \lambda,\ n = 0, 1, 2, \ldots,\ i = 0, 1;$$

$$q_{(n,1)(n-1,1)} = \mu,\ n = 1, 2, 3, \ldots;$$

$$q_{(n,0)(n-1,0)} = \mu_0,\ n = 1, 2, 3, \ldots;$$

$$q_{(n,0)(n,1)} = \theta,\ n = 0, 1, 2, \ldots;$$

$$q_{(n,1)(n,0)} = \zeta,\ n = 0, 1, 2, \ldots.$$

Every customer receives a reward of R units after service. Besides, there exists a waiting cost of C units per time unit that he remains in the system. The server's state is modulated by a random environment $I(t)$, which is a continuous-time Markov chain with state space $\{1, 0\}$. When $I(t) = i$, the system behaves as an $M(\lambda)/M(\mu_i)/1$ queue and $\mu_1 = \mu$. The q-matrix and stationary distribution of $I(t)$ are

$$Q = \begin{pmatrix} -\zeta & \zeta \\ \theta & -\theta \end{pmatrix},\ \pi_1 = \frac{\theta}{\zeta + \theta},\ \pi_0 = \frac{\zeta}{\zeta + \theta}. \quad (1)$$

3. The almost unobservable case

In this case, customers observe $I(t)$, but not $L(t)$. A mixed strategy is specified by the joining probabilities (q_0, q_1). If all customers use the strategy (q_0, q_1), the system is similar to the original queue except that the arrival rate equals λq_i for state i. Let P_{ni} be the stationary distribution of the corresponding system.

Lemma 3.1. *If $\mu > \lambda q_1$ and $\mu_0 > \lambda q_0$, the system is stable.*

Proof. We have the following equations:

$$(\lambda q_0 + \theta)P_{00} = \zeta P_{01} + \mu_0 P_{10},\ (\lambda q_1 + \zeta)P_{01} = \theta P_{00} + \mu P_{11}, \quad (2)$$

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$$(\lambda q_0+\theta+\mu_0)P_{n0}=\lambda q_0 P_{n-1,0}+\zeta P_{n1}+\mu_0 P_{n+1,0},\ n\ge 1, \tag{3}$$

$$(\lambda q_1+\zeta+\mu)P_{n1}=\lambda q_1 P_{n-1,1}+\theta P_{n0}+\mu P_{n+1,1},\ n\ge 1. \tag{4}$$

Starting with $n=0$ and summing these equations, then

$$\mu P_{n+1,1}+\mu_0 P_{n+1,0}=\lambda q_1 P_{n1}+\lambda q_0 P_{n0},\ \ n\ge 0. \tag{5}$$

Note that $\sum_{n=0}^{\infty}P_{ni}=\pi_i$, the quantity π_i is the marginal probability of the external environment $I(t)$ being at state i. π_0 and π_1 are independent of the arrival and service rates. By summing (5) over all n,

$$\mu P_{01}+\mu_0 P_{00}=(\mu-\lambda q_1)\pi_1+(\mu_0-\lambda q_0)\pi_0. \tag{6}$$

From Theorem 1.6 in p. 160 of Anderson [1], since all states are communicating, from the theory of recurrent events, all the probabilities P_{ni} ($n=0,1,2,\ldots,i=0,1$) are either all positive (and sum to one) or, alternatively, all equal to zero. From (6), if $\mu>\lambda q_1$ and $\mu_0>\lambda q_0$, all the probabilities P_{ni} are positive (and sum to one) from the ergodicity theory. The system is stable. □

Let $G_i(z)=\sum_{n=0}^{\infty}P_{ni}z^n$, $i=0,1$. From (2)–(4),

$$[(\lambda q_1 z-\mu)(1-z)+\zeta z]G_1(z)-\theta z G_0(z)=(z-1)\mu P_{01}, \tag{7}$$

$$[(\lambda q_0 z-\mu_0)(1-z)+\theta z]G_0(z)-\zeta z G_1(z)=(z-1)\mu_0 P_{00},$$

$$G_1(z)=\frac{[(\lambda q_0 z-\mu_0)(1-z)+\theta z]\mu P_{01}+\mu_0\theta z P_{00}}{Q(z)}, \tag{8}$$

where $Q(z)=\lambda^2 q_0 q_1 z^3-(\lambda q_0 q_1+q_0\zeta+q_1\theta+q_1\mu_0+q_0\mu)\lambda z^2+(\zeta\mu_0+\theta\mu+\mu\mu_0+\lambda q_1\mu_0+\lambda q_0\mu)z-\mu\mu_0$. From (5), we get

$$(\mu-\lambda q_1 z)G_1(z)+(\mu_0-\lambda q_0 z)G_0(z)=\mu P_{01}+\mu_0 P_{00}. \tag{9}$$

Lemma 3.2. *The function $Q(z)$ has a unique root g in $(0,1)$.*

Proof. If $0<q_i\le 1$, $Q(0)<0$ and $Q(1)>0$. Clearly, $\mu\theta+\mu_0\zeta>\lambda q_1\theta+\lambda q_0\zeta$, if $\frac{\mu}{\lambda q_1}\ge\frac{\mu_0}{\lambda q_0}$, $\frac{\mu}{\lambda q_1}>1$ and $Q(\frac{\mu}{\lambda q_1})=\frac{\mu\zeta}{\lambda q_1}(\mu_0-\frac{\mu q_0}{q_1})\le 0$. The three roots lie in $(0,1)$, $(1,\frac{\mu}{\lambda q_1})$ and $(\frac{\mu}{\lambda q_1},\infty)$. Similarly, if $\frac{\mu_0}{\lambda q_0}>\frac{\mu}{\lambda q_1}$, $\frac{\mu_0}{\lambda q_0}>1$ and $Q(\frac{\mu_0}{\lambda q_0})<0$. When $q_0=q_1=0$, $g=\frac{\mu\mu_0}{\zeta\mu_0+\theta\mu+\mu\mu_0}$. □

The numerator of (8) must equal zero when $z=g$. From (6),

$$P_{01}=\frac{\theta g(\mu\pi_1+\mu_0\pi_0-\lambda q_1\pi_1-\lambda q_0\pi_0)}{\mu\theta g-\mu\varphi(g)}, \tag{10}$$

where $\varphi(g)=(\lambda q_0 g-\mu_0)(1-g)+\theta g$. Since $G_i(1)=\pi_i$, by differentiating (7) and (9) with respect to z and setting $z=1$, then

$$G_0'(1)=\frac{\lambda q_1\pi_1\zeta+\lambda q_0\pi_0\zeta+[\mu P_{01}+(\lambda q_1-\mu)\pi_1](\lambda q_1-\mu)}{\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)}, \tag{11}$$

$$G_1'(1)=\frac{\lambda q_1\pi_1\theta+\lambda q_0\pi_0\theta+[\mu P_{01}+(\lambda q_1-\mu)\pi_1](\mu_0-\lambda q_0)}{\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)}. \tag{12}$$

Due to PASTA property, the probability that there are n customers in the system given that the server is found at state i is

$$p(n|i)=\frac{P_{ni}}{\sum_{k=0}^{\infty}P_{ki}}=\frac{P_{ni}}{\pi_i},\ n=0,1,2,\ldots,\ i=0,1.$$

Let $L_i(q_0,q_1)$ be the conditional queue length in state i, then

$$L_i(q_0,q_1)=\sum_{n=0}^{\infty}np(n|i)=\frac{G_i'(1)}{\pi_i},\ i=0,1.$$

From (1), (10), (11) and (12), we find

$$L_0(q_0,q_1)=\frac{\lambda q_1\theta\zeta+\lambda q_0\zeta^2+\theta(\lambda q_1-\mu)^2}{\zeta[\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)]}+\frac{\theta g(\mu-\lambda q_1)(\lambda q_1\theta+\lambda q_0\zeta-\mu\theta-\mu_0\zeta)}{\zeta(\theta g-\varphi(g))[\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)]}, \tag{13}$$

$$L_1(q_0,q_1)=\frac{\lambda q_1\theta+\lambda q_0\zeta+(\lambda q_1-\mu)(\mu_0-\lambda q_0)}{\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)}+\frac{g(\lambda q_0-\mu_0)(\lambda q_1\theta+\lambda q_0\zeta-\mu\theta-\mu_0\zeta)}{(\theta g-\varphi(g))[\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)]}. \tag{14}$$

In the fully observable case, see Li et al. [4], a joining customer that finds the system at state (n,i) has mean sojourn time $T(n,i)$ and

$$T(n,0)-T(n,1)=\frac{\mu-\mu_0}{\mu_0\zeta+\mu\theta}\left(1-\left(\frac{\mu\mu_0}{\mu\mu_0+\mu_0\zeta+\mu\theta}\right)^{n+1}\right). \tag{15}$$

If a joining customer finds the server at state i, his mean sojourn time is $T(L_i(q_0,q_1),i)$, $T(n,0)$ and $T(n,1)$ are given in Li et al. [4], then

$$T(L_0(q_0,q_1),0)=\frac{(\theta+\zeta)L_0(q_0,q_1)}{\mu_0\zeta+\mu\theta}+\frac{\mu-\mu_0}{\mu\theta+\mu_0\zeta}+\frac{\mu\theta(\mu_0-\mu)}{(\mu\theta+\mu_0\zeta)^2}\left(\frac{\mu\mu_0}{\mu\mu_0+\mu_0\zeta+\mu\theta}\right)^{L_0(q_0,q_1)+1}+\frac{1}{\mu\mu_0+\mu_0\zeta+\mu\theta}\left(\mu_0+\zeta+\theta-\frac{\mu\mu_0^2\zeta(\mu-\mu_0)}{(\mu\theta+\mu_0\zeta)^2}\right), \tag{16}$$

$$T(L_1(q_0,q_1),1)=\frac{\mu_0\zeta(\mu-\mu_0)}{(\mu_0\zeta+\mu\theta)^2}\left(\frac{\mu\mu_0}{\mu\mu_0+\mu_0\zeta+\mu\theta}\right)^{L_1(q_0,q_1)+1}+\frac{1}{\mu\mu_0+\mu_0\zeta+\mu\theta}\left(\mu_0+\zeta+\theta-\frac{\mu\mu_0^2\zeta(\mu-\mu_0)}{(\mu\theta+\mu_0\zeta)^2}\right)+\frac{(\theta+\zeta)L_1(q_0,q_1)}{\mu_0\zeta+\mu\theta}. \tag{17}$$

The expected net reward of such a customer is

$$S_i(q_0,q_1)=R-CT(L_i(q_0,q_1),i),\ \ i=0,1.$$

Lemma 3.3. *If q_1 is fixed, $S_0(q_0,q_1)$ is strictly decreasing for q_0. If q_0 is fixed, $S_1(q_0,q_1)$ is strictly decreasing for q_1.*

Proof. From (13), $\varphi(g)$ is increasing with respect to q_0, the numerator is increasing for q_0 and the denominator is decreasing for q_0. Hence, $L_0(q_0,q_1)$ is increasing for q_0. $T(n,0)$ is increasing with respect to n and thus $S_0(q_0,q_1)$ is decreasing for q_0. Similarly, $S_1(q_0,q_1)$ is strictly decreasing for q_1. □

Lemma 3.4. *If $\mu>\lambda q_1$ and $\mu_0>\lambda q_0$, $S_0(q_0,q_1)<S_1(q_0,q_1)$.*

Proof. From (11) and (12), we get

$$L_0(q_0,q_1)-L_1(q_0,q_1)=\frac{\theta(\mu-\lambda q_1)^2+\zeta(\mu-\lambda q_1)(\mu_0-\lambda q_0)}{\zeta\theta(\mu-\lambda q_1)+\zeta^2(\mu_0-\lambda q_0)}+\frac{[\mu\zeta(\mu_0-\lambda q_0)+\mu\theta(\mu-\lambda q_1)](\theta+\zeta)P_{01}}{\zeta\theta[\theta(\mu-\lambda q_1)+\zeta(\mu_0-\lambda q_0)]}.$$