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A polynomial time algorithm to the economic lot sizing problem with constant capacity and piecewise linear concave costs

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ABSTRACT

It is well-known that the classical economic lot-sizing problem with constant capacity and general concave ordering/inventory cost functions can be solved in $O(T^4)$ time (Florian and Klein, 1971). We show that the problem can be solved in $O(mT^3)$ time when the ordering cost functions are piecewise linear concave and have m line segments with different slopes in a time period in average. Our algorithm makes use of the data structure of range minimum query (RMQ).

1. Introduction

The single-item economic lot-sizing model with constant capacity (ELS-CC) is a classical production planning model. The general ELS-CC problem assumes constant capacity, unrestricted backlogging and general concave ordering, inventory holding and backlogging cost functions. In the literature of lot-sizing research, Florian and Klein [3] have developed an $O(T^4)$ optimal algorithm for the general ELS-CC problem, where T is the number of time periods in the planning horizon. It is still unknown whether the $O(T^4)$ running time complexity of solving the general ELS-CC problem could be reduced. But some direct special cases of ELS-CC can be solved in lower running time complexity. An $O(T^3)$ exact algorithm is presented by van Hoesel and Wagelmans [6] for the special case of ELS-CC when the ordering cost functions are fixed-plus-linear (i.e., a fixed cost is incurred, irrespective of the order size along with variable costs that are proportional with the order size), inventory holding cost functions are linear (but backlogging is not allowed in their model). Later, van Vyve [7] develops an $O(T^3)$ algorithm for the special case of ELS-CC with fixed-plus-linear ordering cost functions and linear inventory holding and backlogging cost functions (backlogging is allowed in his model). Ou [4] presents an $O(T^3)$ algorithm for the special case of ELS-CC with fixed-plus-linear ordering cost functions and general concave inventory holding and backlogging cost functions (backlogging is allowed). In this paper we study the special case of ELS-CC with piecewise linear concave ordering cost functions and general concave inventory holding and backlogging cost functions, where backlogging is allowed. The problem we study is an extension of the model of Ou [4] but still a special case of ELS-CC. We show that the problem can be solved in $O(mT^3)$ time, where m is the average number of line segments with different slopes of the ordering cost function in a time period. Note that in the model of [4], the ordering cost function in each time period has exactly one line segment with a specific slope. As such we generalize the $O(T^3)$ algorithm by Ou [4] which applies to a more restricted class, where the ordering cost functions are fixed-plus-linear, as opposed to our piecewise linear concave ordering cost functions.

In reality, most of the concave ordering cost functions are piecewise linear concave. One of the traditional motivations of piecewise linear concave ordering cost is that the production planner faces multiple competing suppliers, where replenishment from a supplier encounters either high fixed ordering cost but low variable purchasing price, or low fixed ordering cost but high variable purchasing price. The ordering cost function in each period turns out to be piecewise linear concave with m line segments each of which has a different slope if there are m suppliers in the period. Another traditional motivation of piecewise linear concave ordering cost is that the production planner faces only one supplier who provides incremental quantity discount.

Our algorithm is similar to the algorithm presented by Ou [4], but we provide more optimal insights, and make use of range minimum queries (RMQs) to return the minimum of some cost quantities that are predetermined during our backward dynamic programming algorithm, which is the key of achieving the $O(mT^3)$ running time complexity. Given a static array of n totally ordered objects, the range minimum query (RMQ) problem is to build a data structure that allows us to answer efficiently subsequent on-line

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queries of the form "what is the position of a minimum element in the subarray ranging from i to j?" After an $O(n)$-time preprocessing on the n objects, it only takes $O(1)$ time to find out the minimum element in the subarray ranging from i to j once the values of i and j are provided. Furthermore, the space complexity of storing the n objects to support $O(1)$-time RMQ is only $O(n)$. A recent excellent article on RMQ is referred to [2].

2. Model

The economic lot-sizing model we study can be described as follows: There are T time periods in the planning horizon. In each period $i = 1, 2, \ldots, T$, there are a known demand d_i, a given inventory cost function H_i and a given ordering cost function P_i. Let I_i denote the inventory level at end of period i, and X_i the replenishment quantity in period i. It is required that $0 \le X_i \le C$ for $i = 1, 2, \ldots, T$, where C is the given stationary production capacity. Backlogging is allowed. Function H_i is assumed to be general concave over intervals $(-\infty, 0]$ and $[0, +\infty)$, respectively, with $H_i(0) = 0$. Function P_i is assumed to be piecewise linear concave over interval $(0, C]$ with m_i given breakpoints $B_{i,1}, B_{i,2}, \ldots, B_{i,m_i}$, where $0 < B_{i,1} < \cdots < B_{i,m_i} = C$ (for notational simplicity, we define $B_{i,0} = 0$). Specifically, each P_i is fixed-plus-linear over interval $(B_{i,k-1}, B_{i,k}]$ for any $k = 1, 2, \ldots, m_i$, and we can express function P_i as

$$P_i(X_i) = \begin{cases} 0, & \text{if } X_i = B_{i,0}; \\ s_{i,1} + p_{i,1} \cdot X_i, & \text{if } B_{i,0} < X_i \le B_{i,1}; \\ s_{i,2} + p_{i,2} \cdot X_i, & \text{if } B_{i,1} < X_i \le B_{i,2}; \\ \cdots \\ s_{i,m_i} + p_{i,m_i} \cdot X_i, & \text{if } B_{i,m_i-1} < X_i \le B_{i,m_i}; \\ +\infty, & \text{if } B_{i,m_i} < X_i, \end{cases}$$

where

$$0 \le s_{i,1} < s_{i,2} < \cdots < s_{i,m_i},$$

$$p_{i,1} > p_{i,2} > \cdots > p_{i,m_i} \ge 0,$$

and

$$B_{i,k} = \frac{p_{i,k} - p_{i,k+1}}{s_{i,k+1} - s_{i,k}}$$

for $k = 1, 2, \ldots, m_i - 1$. In other words, the curve of P_i over $(0, C]$ is made up of m_i connected line segments, where the kth segment is line $y = s_{i,k} + p_{i,k}x$ over interval $(B_{i,k-1}, B_{i,k}]$, $k = 1, 2, \ldots, m_i$. We let

$$m = \frac{1}{T} \sum_{i=1}^{T} m_i$$

be the average number of line segment with different slopes of an ordering cost function.

We assume that both the initial inventory level at the beginning of period 1 and the inventory level at the end of period T are zero, i.e., $I_0 = I_T = 0$ (we define period 0 to be a dummy period). The problem is to decide the quantities of X_i and I_i $(1 \le i \le T)$ to satisfy the demand in each period, so that the total ordering and inventory cost is minimized. The problem can be formulated as the following mathematical program:

$$\begin{aligned} \mathbf{P}: \quad & \text{minimize} && \sum_{i=1}^{T} \Big[P_i(X_i) + H_i(I_i) \Big] \\ & \text{subject to} && I_i = I_{i-1} + X_i - d_i \quad (i = 1, 2, \ldots, T) \\ &&& I_0 = I_T = 0 \\ &&& 0 \le X_i \le C \quad (i = 1, 2, \ldots, T) \end{aligned}$$

To help the readers easier follow our algorithm, we provide a small example **E** that will be used throughout the paper. Example **E** is as follows: $T = 6$, $(d_1, d_2, d_3, d_4, d_5, d_6) = (4, 12, 1, 8, 5, 4)$, $C = 10$; for $i = 1, \ldots, 6$, $m_i = 2$, $B_{i,1} = 4$; $P_i(X_i) = 1 + 2X_i$ if $0 < X_i \le 4$, $P_i(X_i) = 5 + X_i$ if $4 < X_i \le 10$, $H_i(I_i) = I_i$ if $I_i \ge 0$, and $H_i(I_i) = -2I_i$ if $I_i < 0$.

3. Notation and property

We call period i a *regeneration period* if $I_i = 0$ $(0 \le i \le T)$. Periods 0 and T are both regeneration periods. Each period i is called a *replenishment period* if $X_i > 0$, a *full replenishment period* if $X_i = C$, and a *fractional period* if $0 < X_i < C$.

Let $d_{i,j} = \sum_{r=i}^{j} d_r$ be the cumulative demand in periods $i, i+1, \ldots, j$. Denote $\lceil x \rceil$ as the minimal integer no less than x. For any $1 \le i \le j \le T$, let

$$n_{i,j} = \left\lceil \frac{d_{i,j}}{C} \right\rceil - 1 \quad \text{and} \quad R_{i,j} = d_{i,j} - n_{i,j} \cdot C.$$

For any $1 \le i \le j \le T$, we must have

$$0 < R_{i,j} \le C$$

if $d_{i,j} > 0$. We define $d_{i,j} = n_{i,j} = R_{i,j} = 0$ for any $i > j$.

Denote $\mathbf{P}_0$ as the general problem of ELS-CC, i.e., $\mathbf{P}_0$ is the same as $\mathbf{P}$ except that the ordering cost functions in $\mathbf{P}_0$ are general concave. We give the following property directly, which is well-known in the lot-sizing literature (see, for example, [3]).

Lemma 1. *There exists an optimal solution to $\mathbf{P}_0$ in which for any two consecutive regeneration periods $i-1$ and j $(j \ge i)$, if there are some replenishment periods among $i, i+1, \ldots, j$, then there exists a period $t \in \{i, i+1, \ldots, j\}$ such that*
(i) *no period within $\{i, i+1, \ldots, j\} \setminus \{t\}$ is a fractional period;*
(ii) *the number of full replenishment periods within $\{i, i+1, \ldots, j\} \setminus \{t\}$ is equal to $n_{i,j}$; and*
(iii) *the replenishment quantity in period t equals $R_{i,j}$.*

Note that $\mathbf{P}$ is a special case of $\mathbf{P}_0$. Thus, Lemma 1 is valid to problem $\mathbf{P}$. In our optimal algorithm to $\mathbf{P}$, we only need to consider those optimal solutions satisfying Lemma 1.

We now define two useful cost quantities that will be used in our algorithm. Using the same notations in Ou [4], for any $1 \le i \le j \le T$ and $k = 0, 1, \ldots, j-i+1$, we define $f_{i,j,k}$ as the minimal total cost incurred in periods $i, i+1, \ldots, j$ if $i-1$ is a regeneration period, among $i, i+1, \ldots, j$ no period is a fractional period, and the number of full replenishment periods is equal to k; we also define $\hat{f}_{i,j,k}$ the same as $f_{i,j,k}$ except that j is a regeneration period instead of $i-1$. Let $f_{i,j,k} = \hat{f}_{i,j,k} = +\infty$ if $i > j$ or $k \notin \{0, 1, \ldots, j-i+1\}$. As what has been shown by Ou [4], all of the values in $\{f_{i,j,k}, \hat{f}_{i,j,k} \mid 1 \le i \le j \le T; -T \le k \le T\}$ can be determined recursively in $O(T^3)$ time. In the following analysis we assume that all of those values of $f_{i,j,k}$ and $\hat{f}_{i,j,k}$ have been predetermined.

Let $U_t = \{t, t+1, \ldots, T\}$ for $t = 1, 2, \ldots, T$. For any given $t \in \{1, 2, \ldots, T-1\}$, we sequence the values of $R_{t+1,j}$ for $j = t, t+1, \ldots, T$ such that

$$R_{t+1,\pi_t(t)} \le R_{t+1,\pi_t(t+1)} \le \cdots \le R_{t+1,\pi_t(T)}, \tag{1}$$

where $\pi_t(t), \pi_t(t+1), \ldots, \pi_t(T)$ are a permutation of $t, t+1, \ldots, T$. Notice that $R_{t+1,t} = 0$. By (1), we must have $R_{t+1,\pi_t(t)} = 0$, i.e., $\pi_t(t) = t$.

For any given integers i and t with $1 \le i \le t \le T$, we consider quantity $C - R_{i,t}$, and let $\beta_{i,t} \in \{t, t+1, \ldots, T\}$ denote the largest integer satisfying $R_{t+1,\pi_t(\beta_{i,t})} \le C - R_{i,t}$ (note that $\beta_{i,t}$ must exist since $R_{t+1,\pi_t(t)} = 0 \le C - R_{i,t}$). Provided π_t and $\beta_{i,t}$, we further define

$$A_{i,t,2} = \{j \in U_t \mid R_{t+1,j} \le C - R_{i,t}\} = \{\pi_t(j) \mid j = t, \ldots, \beta_{i,t}\} \tag{2}$$