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Development on the mean inactivity time order with applications

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ABSTRACT

This paper introduces a development of the mean inactivity time order to make stronger stochastic comparisons than the ordinary mean inactivity time order. We apply the new stochastic order to derive some results on Poisson shock models, parallel systems, extreme order statistics and provide some insightful inequalities in the context of a queueing system.

1. Introduction and motivation

Stochastic order relations provide valuable insight into the behavior of complex stochastic systems and enable the user to design and optimize them in order to increase their reliability (cf. Kuo and Rajendra Prasad [13], Baxter and Harche [1], Chern [5], Di Crescenzo and Pellerey [7], Nanda and Hazra [22] and Hazra and Nanda [9]). The applications of stochastic orders and aging notions in analyzing queueing systems have attracted the attention of many researchers (see, e.g., Marshall [18], Bergmann [4], Kuik and Tielemans [12], Müller and Stoyan [20], Szekli [24] and Dewan and Khaledi [6]). Stochastic comparisons of random variables through shifted orders of a special reliability measure have been initiated by Lillo et al. [16] and have been applied to compare reliability systems by Belzunce et al. [3]. They claimed that as the shifted stochastic orders are stronger than the ordinary orders and easily verified in different situations, thus they are useful to figure out which components (and as a result the systems formed from them) have more reliability. In particular, down shifted stochastic orders are tools for detecting the components whose ages have no effect on their operation. In addition, making comparison of a random variable to itself according to down shifted stochastic orders describes some aging phenomena which are quite practical to model the lifetime of products of everyday usage whose failure probabilities are relatively high at the initial stage of their life (cf. Lillo et al. [16] and Shaked and Shanthikumar [23]). Roughly speaking, a down shifted stochastic order stands between the lifetimes of two devices A and B whenever the lifetime of a used device of kind B at any age is stochastically greater than the lifetime of a fresh device of kind A, or vice versa. This guarantees that in spite of any eventual deterioration of the device B with age, a device of kind B at any age is however more reliable than a new device of kind A. A recent reliability measure known as mean inactivity time (MIT), which is also useful in operations research, for example in evaluating mean idle time in a $GI/G/1$ queue (cf. Marshall [18]), has come to use in order to make comparison of lifetime random variables (cf. Kayid and Ahmad [11]).

The purpose of this paper is to introduce down shifted mean inactivity time (DSMIT) order, study some of its properties and then apply it to make comparisons of some reliability models and analyze a $GI/G/1$ queue. The importance of such a study lies on the usefulness of using stronger stochastic comparison based on the MIT function and, as a result, detect those components as well as the systems that have a smaller mean inactivity time function at any stage of their life. In other words, the DSMIT order, in turn, makes a stronger stochastic comparison than the MIT order.

Throughout this paper, X and Y are two lifetime random variables with distribution functions F and G, respectively. Then $X_{(t)} = [t - X|X \leq t]$ and $Y_{(t)} = [t - Y|Y \leq t]$, for $t > 0$, are the inactivity times of X and Y, respectively, at the time point t. Notice that $X_{(t)}$ and $Y_{(t)}$ have respective distributions functions $F_{(t)}$ and $G_{(t)}$, which are given by $F_{(t)}(a) = (F(t) - F(t-a))/F(t)$ and $G_{(t)}(a) = (G(t) - G(t-a))/G(t)$ for $0 \leq a \leq t$, while $F_{(t)}(a) = G_{(t)}(a) = 0$ for all $a < 0$ and $F_{(t)}(a) = G_{(t)}(a) = 1$ for all $t > a$. For any $t > 0$, the MIT functions of X and Y are defined by $m_X(t) = E\left[X_{(t)}\right]$ and

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$m_Y(t) = E\left[Y_{(t)}\right]$, given by $m_X(t) = \int_0^t F(x)dx/F(t)$ and $m_Y(t) = \int_0^t G(x)dx/G(t)$, respectively. The MIT order is defined as follows (cf. Kayid and Ahmad [11]).

Definition 1.1. The lifetime random variable X is said to be smaller than the lifetime random variable Y in the MIT order (denoted by $X \le_{MIT} Y$) if $m_X(t) \ge m_Y(t)$ for all $t > 0$ or equivalently, if $\int_0^t G(x)dx / \int_0^t F(x)dx$ is increasing in $t > 0$.

Definition 1.2. Suppose that the nonnegative random variable Y denotes the lifetime of a new unit with survival function $\bar{G} = 1 - G$. Then the random variable Y_t having survival function $\bar{G}_t(a) = \bar{G}(t+a)/\bar{G}(t)$ for $a > 0$, represents the lifetime of the used unit of age $t > 0$.

The paper is organized as follows. In Section 2, the basic definition of the DSMIT order and some of its elementary properties are given. In Section 3, preservation of the DSMIT order under shock models is discussed. In Section 4, the reversed preservation property of the DSMIT order for parallel systems is obtained. The DSMIT order between two extreme order statistics is provided in Section 5. Finally, in Section 6, the DSMIT order is applied to the $GI/G/1$ queue.

2. Down shifted mean inactivity time order

Definition 2.1. The lifetime random variable X is smaller than the lifetime random variable Y in the DSMIT order (denoted as $X \le_{MIT\downarrow} Y$), if $X \le_{MIT} Y_t$, for all $t \ge 0$.

From Definition 1.2, the DSMIT order states that the MIT of a new device with lifetime X is entirely greater than the MIT of a used device at any age whose fresh version has original lifetime Y. In other words, if X and Y are lifetimes of two devices A and B, respectively, satisfying the DSMIT order then the inactivity time of the device B without considering its age is less (in mean) than the inactivity of a new device of kind A. This means that the device B has a rather good operation in comparison with the device A as it is expected from the device B of an unknown age to have a smaller MIT function. The following example explains the concept. Note that, the DSMIT order is stronger than the MIT order (cf. Kayid and Ahmad [11]):

$X \le_{MIT\downarrow} Y \;\Rightarrow\; X \le_{MIT} Y$.

Example 2.1. Suppose that X and Y are two nonnegative random variables with densities $f(x) = e^{-x},\ x \ge 0$ and $g(y) = ye^{-y},\ y \ge 0$, respectively. One can easily see that $X \le_{MIT\downarrow} Y$.

Example 2.2. For a random variable Y with exponential distribution, it is easily seen that for any lifetime random variable X, $X \le_{MIT\downarrow} Y$ if, and only if, $X \le_{MIT} Y$.

As the following counterexample clarifies, the MIT order does not imply the DSMIT order.

Counterexample 2.1. Let X and Y have distribution functions $F(x) = x^2,\ 0 \le x \le 1$ and $G(y) = y^3,\ 0 \le y \le 1$, respectively. Then we see that $m_X(t) = t/3$ and that $m_Y(t) = t/4$. Thus $X \le_{MIT} Y$. Further, $m_{Y_t}(x) = (x^4 + 4tx^3 + 6t^2x^2)/(4x^3 + 12tx^2 + 12t^2x)$ for $x \ge 0$ and $t \ge 0$. It can be observed that if $t \to \infty$, then $m_X(x) \not\ge m_{Y_t}(x)$ for all $x \in [0, 1]$. Hence, $X \not\le_{MIT\downarrow} Y$.

The next result presents a condition under which the MIT order implies the DSMIT order. The proof is straightforward and hence we omit it.

Theorem 2.1. *If $Y \le_{MIT} Y_t$, for all $t \ge 0$, then $X \le_{MIT} Y$ implies $X \le_{MIT\downarrow} Y$.*

The following example illustrates an applications of Theorem 2.1.

Example 2.3. Suppose that X and Y are two nonnegative random variables with densities $f(x) = e^{-\frac{1}{2x}}/(2x^2(1 - e^{-\frac{1}{2}})),\ x \ge 1$ and $g(y) = e^{-\frac{1}{y}}/(y^2(1 - e^{-1})),\ y \ge 1$, respectively. It can be verified that $X \le_{MIT} Y$ and that Y possesses the property $Y \le_{MIT} Y_t$, for all $t \ge 0$. Hence, by Theorem 2.1, $X \le_{MIT\downarrow} Y$.

3. Poisson shock models

Suppose that a system has an ability to survive a random number of shocks, denoted by N, so that N and the random inter-arrival times between the $(j-1)$th and jth shocks, denoted by X_j, are independently distributed. Then the lifetime of the system is exposed as $X = \sum_{j=1}^{N} X_j$ stating that shock models are particular random sums. Now, we look at a special case where the inter-arrivals are independent with exponential distribution (with mean $1/\lambda$), then it is proved that X has distribution

$$F(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} P_k\,,\ t \ge 0,$$

where $P_k = P[N \le k]$ for all $k \in \mathbb{N}$ (and $\bar{P}_0 = 1$). This kind of shock models are called Poisson shock models that have been studied considerably in the literature (see, e.g., Belzunce et al. [2] and Finkelstein [8]).

Suppose two devices are subjected to Poisson shocks and let N and M be the random number of shocks survived by these devices. Moreover, denote by P_k, and Q_k respective distributions of N and M, which are indeed the probabilities of surviving the first k shocks. Let X and Y denote the lifetime of the first and the second devices, respectively, which have distributions

$$F(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} P_k, \quad t \ge 0, \tag{3.1}$$

and

$$G(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} Q_k, \quad t \ge 0. \tag{3.2}$$

In the sequel, we provide some sufficient conditions under which the DSMIT order between N and M is translated to the DSMIT order between the lifetimes X and Y. Before that, we must introduce two concepts.

Definition 3.1. Let N and M be two positive integer-valued random variables with distributions P_k and Q_k, respectively. Then N is said to be smaller than M in discrete down mean inactivity time order (written as $N \le_{d\text{-}MIT\downarrow} M$), whenever

$$\frac{\sum_{k=0}^{i-1}(Q_{j+k} - Q_j)}{\sum_{k=0}^{i-1} P_k},$$

is non-decreasing in $i \ge 1$, for all $j \in \mathbb{N}$.

In the sequel, we need the following notion from Karlin [10].

Definition 3.2. The function $h : (t, s) \to h(t, s)$ is called totally positive (reverse regular) of order two, abbreviated by TP_2 (RR_2), in $(t, s) \in \mathcal{T} \times \mathcal{S}$, if $h(t_1, s_2)h(t_2, s_1) \le (\ge)\, h(t_1, s_1)h(t_2, s_2)$, for all $t_1 \le t_2 \in \mathcal{T}$ and $s_1 \le s_2 \in \mathcal{S}$, where $\mathcal{T}$ and $\mathcal{S}$ are two real subsets of $\mathbb{R}$.

The following result states that if the random number of shocks that the device can survive, is increased with respect to the DSMIT order, then the lifetime of the device is stochastically increased in terms of the DSMIT order.