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On the optimality conditions of a price-setting newsvendor problem

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ABSTRACT

We analyze a price-setting newsvendor problem with an additive–multiplicative demand. We show that the unimodality of the newsvendor profit function holds when the underlying random term has an increasing failure rate and the demand functions satisfy certain concavity conditions. Furthermore, we show that the optimal price decreases in the order quantity. Finally, we compare our optimality conditions with those existing in the literature.

1. Introduction

Integrating pricing and inventory replenishment decisions under demand uncertainty has proved to be a successful operations strategy of many firms including Amazon, Dell, Walmart and J.C. Penney [4,6]. By adopting proactive pricing, firms can better match supply with demand, which leads to significant profit increases. The benefits of jointly deciding pricing and inventory replenishment level have also been well documented in various academic studies [5,8,9].

The building block for joint inventory and pricing decisions research is the newsvendor model with pricing. The major difficulty in studying this problem is to establish concavity or unimodality of the profit function. Although many analytical results on the optimality conditions have been developed in recent years [10,12,13,17], the existing literature often makes strong assumptions on the demand function forms and demand uncertainty distributions, such as one parameter additive or multiplicative only demand model and specific uncertainty distributions. These assumptions simplify analysis, but their limitations in capturing reality limit their applicability in practice.

To address this issue, we use a general additive–multiplicative demand model to analyze the problem of joint inventory and pricing decisions. To derive the optimality conditions, we make two assumptions: (1) the random term in the demand has an increasing failure rate (IFR); (2) the demand function satisfies certain concavity conditions. [12] studies the same problem by first solving for the optimal order quantity and then finding the optimal price. It identifies three conditions to be met for establishing the optimality of the profit function: (1) the riskless profit function is log-concave; (2) the coefficient of variation is log-convex and (3) the distribution of the random term satisfies a specific condition. [10] introduces the concept of lost-sale elasticity. Assuming the random term has IFR distribution, it shows that when the lost-sale elasticity satisfies certain conditions, the concavity or unimodality of the profit function is implied. [14] conducts a similar analysis of the problem from the price elasticity view point. It shows that when both price elasticities of the location and scale parameters in demand are increasing in price, and the elasticity of the location parameter increases faster than the price elasticity of the scale parameter, the unimodality of the profit function is obtained. [1] analyzes a risk-averse price-dependent newsvendor and shows the concavity of the profit functions for additive and multiplicative demands. A focus of the paper is to elaborate the difference in pricing and ordering behaviors of the risk averse and risk neutral newsvendors. [17] obtains the unimodality when the random term has log-concave distribution and the demand functions satisfy certain conditions. Compared with the aforementioned studies, our proof is new and the resulting optimality conditions are different. As our analysis unfolds, we show that our optimality conditions and the conditions obtained in the existing literature do not imply each other. In fact, our result complements the existing results.

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The rest of this paper is organized as follows. Section 2 presents the newsvendor pricing model and our analytical results. We discuss our results by comparing them with the existing literature in Section 3.

2. Model and analysis

In this section, we present the price-setting newsvendor model and derive conditions for the optimality of the firm's expected profit function.

2.1. Model

A risk neutral firm buys a product at a unit cost c and sells the product to customers at a retail price $p \in [\underline{p}, \overline{p}]$ over a single selling season. The demand during the selling season depends on the retail price p and is random. Let $D(p, \epsilon)$ denote the price-dependent demand, where ϵ is a random variable. The firm simultaneously decides the retail price p and the order quantity y at the beginning of the selling season before observing the demand. After the demand materializes, the firm satisfies the demand with the product's available stock. If the firm does not have enough stock, that is, $y \leq D(p, \epsilon)$, the unsatisfied demand is lost with no penalty for lost sales. The unsold inventory, if any, is salvaged at zero value. Note that our analysis can be extended easily to the cases of nonzero penalty and nonzero salvage value.

As in [7,8,12,17], we consider the additive–multiplicative demand model

$$D(p, \epsilon) = \mu(p) + \sigma(p)\epsilon, \tag{1}$$

where $\mu(\cdot) \geq 0$ and $\sigma(\cdot) \geq 0$ are deterministic functions of price p, and ϵ is a nonnegative random variable with mean $\tilde{\mu}$ and standard deviation $\tilde{\sigma}$. This demand model generalizes the widely used multiplicative or additive only demand models. In our generalized model, the retail price can have different effects on the location and scale parameters of the demand. Whereas, in the multiplicative only models, the price has the same effects on the demand mean and standard deviation, i.e., the coefficient of variation is independent of price.

Let $f(\cdot)$ and $F(\cdot)$ denote the density and cumulative distribution functions of ϵ, respectively. In addition, let $\bar{F}(\cdot) = 1 - F(\cdot)$. We define $G(\cdot|p)$ as the conditional distribution of $D(p, \epsilon)$ for a given retail price p. Thus, $G(x|p) = F\left(\frac{x-\mu(p)}{\sigma(p)}\right)$. The failure rate of ϵ is defined as $h(\cdot) = f(\cdot)/\bar{F}(\cdot)$.

For a given retail price p and order quantity y, the firm's expected profit (profit, hereafter) is

$$\pi(y, p) = p\,\mathrm{E}\min\{y, \mu(p) + \sigma(p)\epsilon\} - cy. \tag{2}$$

The firm determines $y \in [0, \infty)$ and $p \in [\underline{p}, \overline{p}]$ to maximize its profit, that is $\max_{y,p} \pi(y, p) = p\,\mathrm{E}\min\{y, \mu(p) + \sigma(p)\epsilon\} - cy$. Unfortunately, the concavity of the profit function does not hold under general conditions. Thus, researchers have tried to establish conditions under which the profit function is quasi-concave (i.e., unimodal) or log-concave (since the log-concavity guarantees unimodality [12]).

2.2. Analysis of the optimality conditions

To derive the optimality conditions, we make two assumptions: (i) the random term in the demand has an increasing failure rate (IFR); (ii) the demand function satisfies certain concavity conditions.

Assumption (i) the distribution of ϵ has an increasing failure rate (IFR), that is, $f(\cdot)/\bar{F}(\cdot)$ is an increasing function.

Assumption (ii) the functions $\mu(p)$ and $\sigma(p)$ are twice continuously differentiable and strictly decreasing in price p. In addition, $p\mu(p)$ and $p\sigma(p)$ are concave in p.

Most distributions commonly used in the operations management literature, such as normal, gamma, uniform and logistic distributions have IFR; see [3,10,11,15]. As our analysis unfolds, we show that the IFR distribution assumption leads to a new set of optimality conditions for the unimodality of the profit function. The assumption of the demand to be decreasing in price is standard. The concavity assumption implies that the revenue function is concave in price, e.g., marginal revenue is decreasing in retail price. It is also a widely adopted assumption in the operations management literature [10,18].

To facilitate analysis, we rewrite the profit function as

$$\begin{aligned}\pi(y, p) &= p\mathrm{E}\min\{y, \mu(p) + \sigma(p)\epsilon\} - cy \\ &= p\sigma(p)\mathrm{E}\min\{y/\sigma(p), \kappa(p) + \epsilon\} - cy \\ &= p\sigma(p)S(y, p) - cy,\end{aligned} \tag{3}$$

where $\kappa(p) = \mu(p)/\sigma(p)$ and $S(y, p) = \mathrm{E}\min\{y/\sigma(p), \kappa(p) + \epsilon\}$. For ease of exposition, define $\Theta(x) = \int_x^\infty (u - x)f(u)du$. The profit function can then be written as

$$\pi(y, p) = p\sigma(p)\big\{\kappa(p) + [\tilde{\mu} - \Theta(y/\sigma(p) - \kappa(p))]\big\} - cy. \tag{4}$$

Note that $[\tilde{\mu} - \Theta(y/\sigma(p) - \kappa(p))] = \int_0^{y/\sigma(p)-\kappa(p)} \bar{F}(t)dt \geq 0$. Since ϵ is a nonnegative random variable, for any given p and y, we have $y \geq \mu(p)$. Thus, $y/\sigma(p) - \kappa(p) \geq 0$.

Define $V(z) = \frac{\int_0^z tf(t)dt}{[\tilde{\mu}-\Theta(z)]}$ and $U(z) = \frac{F(z)}{[\tilde{\mu}-\Theta(z)]}$. To establish the optimality conditions for the profit function, we need the monotone property of $V(z)$ and $U(z)$ stated in the following lemma.

Lemma 1 (*Monotone Property*). *Under Assumptions* (i) *and* (ii), *both $V(z)$ and $U(z)$ are nondecreasing functions of z and $0 \leq V(z) < 1$.*

Proof of Lemma 1. Under Assumption (i), the distribution of ϵ has an IFR, so it also has an increasing generalized failure rate (IGFR). It follows that $V(z)$ is nondecreasing in z, which is proved in [15]. Since $V'(z) \geq 0$ and $\lim_{z\to\infty} V(z) = \frac{\int_0^\infty tf(t)dt}{\int_0^\infty \bar{F}(t)dt} = 1$, we have $0 \leq V(z) < 1$.

Next we prove the monotone property of $U(z)$. Taking the first derivative of $U(z)$, we have

$$\begin{aligned}U'(z) &= \frac{f(z)[\tilde{\mu} - \Theta(z)] - F(z)\bar{F}(z)}{[\tilde{\mu} - \Theta(z)]^2} \\ &= \frac{f(z)\int_0^z \bar{F}(t)dt - F(z)\bar{F}(z)}{[\tilde{\mu} - \Theta(z)]^2} \\ &= [\bar{F}(z)]\frac{h(z)\int_0^z \bar{F}(t)dt - F(z)}{[\tilde{\mu} - \Theta(z)]^2}.\end{aligned} \tag{5}$$

Define $\triangle(z) = h(z)\int_0^z \bar{F}(t)dt - F(z)$. We have $\lim_{z\to 0} \triangle(z) = 0$. Further $\triangle'(z) = h'(z)\int_0^z \bar{F}(t)dt \geq 0$, and therefore, when the distribution of ϵ has an IFR, we have $U'(z) \geq 0$. This completes the proof.

We are now ready to present the optimality conditions.

Theorem 1 (*Optimality Conditions*). *Under Assumptions* (i) *and* (ii), *there exists a unique maximizer $(\hat{y}, \hat{p})$ which maximizes the firm's expected profit function. Furthermore, the optimal price $\hat{p}(y)$ for any given order quantity y is decreasing in y.*