Operations Research Letters 44 (2016) 702–705

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

On linearization techniques for budget-constrained binary quadratic programming problems

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ARTICLE INFO

Article history:
Received 1 February 2016
Received in revised form
2 September 2016
Accepted 2 September 2016
Available online 10 September 2016

Keywords:
Binary quadratic programming
Budget constraint
Linearization
Linear programming relaxation

ABSTRACT

Glover's linearization technique is revisited for solving the binary quadratic programming problem with a budget constraint (BBQP). When compared with the recent two linearizations for (BBQP), it not only provides a tighter relaxation at the root node, but also has a much better computational performance for globally solving (BBQP).

1. Introduction

The budget-constrained binary quadratic programming problem (BBQP) is to minimize a general quadratic function with binary variables and an additional budget constraint. It can be formulated as follows:

$$\min\ x^TQx = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_ix_j \tag{1}$$

$$\text{s.t.}\ \ x \in X := \left\{x | e^Tx = p, x \in \{0,1\}^n\right\}, \tag{2}$$

where $Q = (q_{ij})$ is an $n \times n$ symmetric matrix, $e = (1, \ldots, 1)^T$ and $p \in \{1, \ldots, n\}$ is a given integer budget. It could be assumed that $p \leq n/2$, since otherwise we can replace all the variables x_i $(i = 1, \ldots, n)$ with $y_i := 1 - x_i$.

(BBQP) is NP-hard, as its special applications include the p-dispersion-sum problem [4,9,14] in facility location, the dense k-subgraph problem [5] and the p-clique problem [11,12] in graph theory.

The linearization techniques for solving binary quadratic programming problems were first introduced by Zangwill [18] and Watters [16], and then advanced by many researchers, see for example, Glover and Woolsey [7,8], Glover [6], Adams and Sherali [1,2] and Sherali and Smith [15]. For (BBQP), Chaovalitwongse et al. [3] proposed a linearization technique. Recently, it has been improved by He et al. [10].

Chaovalitwongse et al.'s linearization (CL) is equivalent to (BBQP) when Q is a nonnegative matrix. However, we prove in this paper that the linear programming lower bound of (CL) is always zero. Then, we use a counterexample to show that the improved version of (CL), He et al.'s linearization (HL), failed to be equivalent to (BBQP) when Q is not nonnegative. Applying Glover's linearization scheme to (BBQP) yields a new linearization, denoted by (GL). It is always equivalent to (BBQP) without any assumption on Q. For the quadratic assignment problem, the corresponding linearization is also known as Xia–Yuan linearization [17]. Compared with the two recent linearizations (CL) and (HL), our new linearization (GL) not only provides a tighter relaxation at the root node, but also has a much better numerical performance for globally solving (BBQP).

The remainder is organized as follows. Section 2 studies the two existing linearization techniques. In Section 3, we propose a new linearization method. Section 4 presents the numerical comparison.

Notations Let $v(\cdot)$ be the optimal value of $(\cdot)$. conv(X) is the convex hull of X. R$(\cdot)$ is the linear programming relaxation of $(\cdot)$. For a matrix $Q = (q_{ij})$, $\|Q\|_\infty = \max_{i\in\{1,\ldots,n\}} \sum_{j=1}^{n} |q_{ij}|$. $Q \geq 0$ means that $q_{ij} \geq 0$ for all i and j. Notation $1 : n$ stands for $1, \ldots, n$.

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http://dx.doi.org/10.1016/j.orl.2016.09.002
0167-6377/

2. Two existing linearization techniques for (BBQP)

We first present and then study the two existing linearization techniques for (BBQP) due to Chaovalitwongse et al. [3] and He et al. [10], respectively.

Chaovalitwongse et al. [3] proposed the following linearization for (BBQP), denoted by (CL):

$$\begin{aligned}
&\min \sum_{i=1}^{n} s_i \\
&\text{s.t. } x \in X, && (3)\\
&\qquad \sum_{j=1}^{n} q_{ij}x_j - y_i - s_i = 0, \quad i = 1:n, && (4)\\
&\qquad y_i \le \mu(1-x_i), \quad i = 1:n, && (5)\\
&\qquad y_i \ge 0, \quad s_i \ge 0, \quad i = 1:n, && (6)
\end{aligned}$$

where $\mu = \|Q\|_\infty$. Under the assumption $Q \ge 0$, it was proved that $v(\text{CL}) = v(\text{BBQP})$ [3]. However, as shown in the following, under the assumption $n \ge 2p$, the linear programming relaxation of (CL) at the root node always gives a trivial lower bound: zero.

Theorem 2.1. *Suppose $Q \ge 0$ and $n \ge 2p$. Then, $v(\text{R(CL)}) = 0$.*

Proof. Let $\widetilde{x} = \left(\frac{p}{n}, \dots, \frac{p}{n}\right)^T, \widetilde{s} = (0, \dots, 0)^T$ and $\widetilde{y}_i = \frac{p}{n}\sum_{j=1}^{n} q_{ij} \ge 0$ for $i = 1:n$. It is sufficient to show that $(\widetilde{x}, \widetilde{y}, \widetilde{s})$ is a feasible solution of R(CL), which is obtained by replacing the constraint (3) with $x \in \text{conv}(X)$. The inequality (4) trivially holds at $(\widetilde{x}, \widetilde{y}, \widetilde{s})$. The inequality (5) holds since

$$\begin{aligned}
\widetilde{y}_i - \mu(1-\widetilde{x}_i) &= \frac{p}{n}\left(\sum_{j=1}^{n} q_{ij} - \frac{n-p}{p}\mu\right)\\
&\le \frac{p}{n}\left(\sum_{j=1}^{n} q_{ij} - \mu\right) \le 0,
\end{aligned}$$

where the two inequalities follow from the assumption $n \ge 2p$ and the definition of μ, respectively. The proof is complete. □

Recently, He et al. [10] improved (CL) to the following linearization, denoted by (HL):

$$\begin{aligned}
&\min \sum_{i=1}^{n} (s_i - \sigma x_i) \\
&\text{s.t. } x \in X, \\
&\qquad \sum_{j=1}^{n} q_{ij}x_j - y_i - s_i + \sigma = 0, \quad i = 1:n, \\
&\qquad y_i \le \mu(1-x_i), \quad i = 1:n, \\
&\qquad y_i \ge 0, \quad s_i \ge 0, \quad i = 1:n,
\end{aligned}$$

where μ and σ are constants: $\mu = \|Q\|_\infty$ and $\sigma = \mu - \max_{i=1:n}\max_{x\in X}\{\sum_{j=1}^{n} q_{ij}x_j\}$. It was concluded in corollary 1 in [10] that $v(\text{BBQP}) = v(\text{HL})$. However, this is true only when $Q \ge 0$. Below is a counterexample.

Let $n = 4$, $p = 2$ and

$$Q = \begin{bmatrix} 0 & -2 & 0 & 0\\ -2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We have $v(\text{BBQP}) = -4 < v(\text{HL}) = 2$.

3. Glover's linearization for (BBQP)

Based on Glover's linearization scheme, we propose a "new" linearization for (BBQP). We first reformulate the objective function (1) as:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}x_ix_j = \sum_{i=1}^{n}\left(\left(\sum_{j\ne i} q_{ij}x_j\right)x_i + q_{ii}x_i\right).$$

Then, it is not difficult to verify that the equality

$$\left(\sum_{j\ne i} q_{ij}x_j\right)x_i = \max\left\{l_ix_i,\, u_ix_i - u_i + \sum_{j\ne i} q_{ij}x_j\right\}$$

holds for any $x \in X$, where

$$u_i = \max_{x\in X}\sum_{j\ne i} q_{ij}x_j = \sum_{j=1}^{p} q_{i\varphi_i(j)},$$

$$l_i = \min_{x\in X,\ x_i=1}\sum_{j\ne i} q_{ij}x_j = \sum_{j=1}^{p-1} q_{i\phi_i(j)},$$

φ_i be the permutation of $\{1:n\}$ such that $q_{i\varphi_i(1)} \ge q_{i\varphi_i(2)} \ge \cdots \ge q_{i\varphi_i(n)}$ (q_{ii} is redefined as 0), and ϕ_i be the permutation of $\{1,\dots,i-1,i+1,\dots,n\}$ such that $q_{i\phi_i(1)} \le q_{i\phi_i(2)} \le \cdots \le q_{i\phi_i(n)}$.

For $i = 1:n$, introducing the new variable z_i to replace $(\sum_{j\ne i} q_{ij}x_j)x_i$ yields the following "new" linearization, denoted by (GL):

$$\begin{aligned}
&\min \sum_{i=1}^{n} (z_i + q_{ii}x_i) \\
&\text{s.t. } x \in X, \\
&\qquad z_i \ge l_ix_i, \quad i = 1,\dots,n, && (7)\\
&\qquad z_i \ge u_ix_i - u_i + \sum_{j\ne i} q_{ij}x_j, \quad i = 1:n && (8)
\end{aligned}$$

and the following result on the equivalence:

Theorem 3.1. $v(\text{GL}) = v(\text{BBQP})$.

Next, we can show that the linear programming relaxation of (GL) in general is at least as tight as that of (HL) at the root node.

Theorem 3.2. *When $Q \ge 0$, $v(\text{R(GL)}) \ge v(\text{R(HL)})$.*

Proof. Setting $s_i = z_i + \sigma x_i + q_{ii}x_i$ in (HL) yields the equivalent formulation:

$$\begin{aligned}
&\min \sum_{i=1}^{n} (z_i + q_{ii}x_i) \\
&\text{s.t. } x \in X, \\
&\qquad z_i \ge -\sigma x_i - q_{ii}x_i, \quad i = 1:n, && (9)\\
&\qquad z_i \le \sum_{j\ne i} q_{ij}x_j + \sigma - \sigma x_i, \quad i = 1:n, && (10)\\
&\qquad z_i \ge \sum_{j\ne i} q_{ij}x_j + (\mu-\sigma)(x_i-1), \quad i = 1:n. && (11)
\end{aligned}$$

Notice that R(GL) and R(HL) are obtained by replacing $x \in X$ with $x \in \text{conv}(X)$ in (GL) and (HL), respectively. Let (x, z) be any feasible solution of R(GL). Then, either the inequality (9) or the inequality (11) is active at (x, z). Since $Q \ge 0$, $\sigma \ge 0$ and $0 \le x_i \le 1$ for $i = 1:n$, for the right-hand side of (9), we have

$$-\sigma x_i - q_{ii}x_i \le -\sigma x_i \le \sum_{j\ne i} q_{ij}x_j + \sigma - \sigma x_i,$$