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A logarithmic approximation for polymatroid congestion games

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ABSTRACT

We study the problem of computing a social optimum in polymatroid congestion games, where the strategy space of every player consists of a player-specific integral polymatroid base polyhedron on a set of resources. For non-decreasing cost functions we devise an H_ρ-approximation algorithm, where ρ is the sum of the ranks of the polymatroids and H_ρ denotes the ρ-th harmonic number. The approximation guarantee is best possible up to a constant factor and solves an open problem of Ackermann et al. (2008).

1. Introduction

Congestion games have become a standard game-theoretic model describing the allocation of exhaustible resources by selfish players. In the basic model of Rosenthal [21], there is a finite set of players and resources and each player is associated with a set of allowable subsets of resources. A pure strategy of a player consists of an allowable subset. Congestion on a resource is modeled by a load-dependent cost function which is usually non-decreasing and solely depends on the number of players using the resource. In the context of *network games*, the resources may correspond to edges of a graph, the allowable subsets correspond to the simple paths connecting a source and a sink and players choose minimum cost paths. Rosenthal proved in his seminal paper that congestion games always admit a pure Nash equilibrium.

In this paper, we focus on so-called *polymatroid congestion games*, where the strategy space of every player consists of the set of vectors in a player-specific integral polymatroid base polyhedron defined on the ground set of resources. This can be viewed as a game in which every player chooses a multiset of the resources rather than a subset. These games have numerous applications as they include for instance matroid congestion games, singleton congestion games (rank-1 matroids) or spanning tree congestion games, where every player selects a spanning tree of a player-specific subgraph of a given graph. We consider the problem of computing a minimum cost solution in polymatroid congestion games.

1.1. Applications in optimization

Computing a minimum cost solution over an integral polymatroid base polytope is a core problem in combinatorial optimization that has received considerable attention in the past, cf. [14,11,29]. These works pose quite restrictive assumptions on the used cost functions, namely, that they either have fixed-cost structure [29], or are separable convex [14,11].

For several practical applications, however, these assumptions do not apply. Consider the *tree packing problem* in network design, where there is an undirected graph $G = (V, E)$ with non-negative and non-decreasing edge cost functions $c_e(\ell), e \in E$. The goal is to compute integral capacities on edges so that we can serve n minimum spanning trees in G with minimum cost. In contrast to previous works, by allowing arbitrary non-decreasing cost functions we are able to model more realistic cost functions occurring in practice. Typical cost functions are step functions, where every cost level corresponds to a different cable type that can be installed (cf. [3]). Andrews et al. [2], for instance, discussed telecommunication network design problems, where cost functions with "diseconomies of scale" are used to model the cost of energy consumption when routers apply speed scaling to process packets. The used cost function has the form $c_r(\ell) = \sigma + \delta \cdot \ell^\alpha, \alpha > 1, \sigma, \delta > 0$, if $\ell > 0$ and $c_r(0) = 0$. Clearly, this function and also the previous function are neither concave nor convex.

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1.2. Applications in game theory

The problem of computing minimum cost solutions in polymatroid congestion games is important for scenarios where a central planner can implement a solution or when players collaborate. Additionally, minimum cost solutions serve as building blocks for other cost-efficient solutions, e.g., as in [27], where a minimum cost solution is used for defining cost sharing protocols with low price of stability/anarchy. In fact, Ackermann et al. [1, Section 2.2] as well as von Falkenhausen and Harks [27, Section 5] state as an open problem to characterize the computational complexity of computing a minimum cost solution in matroid congestion games with non-decreasing cost functions, which is a special case of the problem we consider in this paper (cf. Remark 2). However, in the case computing an optimal solution is NP-hard, one of the common approaches is to find approximation algorithms. Meyers and Schulz [19] investigated the limitations of such an approach by studying the inapproximability of several of these problems. We complement this type of research by showing a logarithmic approximation that is best possible up to a constant factor.

1.3. Our results

We devise an H_ρ-approximation algorithm, where ρ is the sum of the ranks of the player-specific polymatroids (the value of the polymatroid function on the entire set of resources) and H_ρ denotes the ρ-th harmonic number. This approximation guarantee is best possible (up to a constant factor) for algorithms with polynomial running time, unless NP $\subseteq$ TIME$(n^{\mathcal{O}(\log\log n)})$. As a byproduct we show that matroid congestion games are also H_ρ-approximable in polynomial time, a result that partially settles an open problem of [1,27]. For matroid congestion games, we only leave a gap if ρ is not polynomially bounded in the number of players.

1.4. Literature review

Computing social optima. Werneck et al. [28] studied the complexity of computing a social optimum in spanning tree congestion games. For convex cost functions they devised an efficient algorithm computing an optimal solution. Essentially, convex cost functions allow to linearize the cost function and then to apply a greedy algorithm. Ackermann et al. [1] extended the work of [28] by observing that the same idea is applicable to matroid congestion games still requiring that cost functions are convex. For spanning tree games with non-monotonic cost functions, they showed that computing a social optimum is NP-hard. The case of matroid congestion games with general non-decreasing cost functions is posed as an open problem [1, Section 2.2].

It should be noted that the positive results for convex cost functions were already implied by previous works, perhaps not so obvious. Groenevelt [14] and Fujishige [11] presented polynomial time algorithms to minimize a convex separable function over an integral polymatroid base polyhedron. Since the matroid rank function is submodular, the strategy space for every player can equivalently be represented as an integral polymatroid base polyhedron. Using that the sum of polymatroid base polyhedra is again a polymatroid base polyhedron, the results of Groenevelt [14] and Fujishige [11] thus already imply a polynomial time algorithm for computing a social optimum for matroid congestion games with convex cost functions. For polymatroids with fixed costs for all resources, Wolsey [29] showed that the greedy algorithm gives a logarithmic approximation. In contrast to these works, we consider the case of arbitrary non-decreasing cost functions.

A special case of polymatroid congestion games is that of singleton congestion games with arbitrary non-decreasing cost functions. Harks and von Falkenhausen [17] devised an H_n-approximation algorithm for the social cost, where n is the number of players. Their algorithm is based on successive network flow computations on a suitably defined capacitated graph. Moreover, they showed this result is essentially best possible up to a constant factor, as they show that this optimization problem is hard to approximate within a factor of $c \log n$ for any $c < 1$, a reduction we will extend for our hardness result (cf. Lemma 1).

Meyers and Schulz [19] classified the complexity of computing a social optimum for general congestion games as well as network congestion games and differentiated between asymmetric and symmetric strategy spaces. In the case of network congestion games, they also distinguished the case in which all players share a common source. Regarding the cost functions, they differentiated between five types: non-decreasing, convex non-decreasing, non-increasing, concave non-increasing, and non-monotonic cost functions. For all combinations of strategy spaces and cost functions they established the complexity of finding the social optimum. Most of the resulting problems are inapproximable to any finite factor. In particular, the asymmetric case with non-decreasing costs is not approximable to any finite factor. Very recently, Roughgarden [22] studied the impact of the computational complexity of computing socially optimal solutions on the price of anarchy. He derived a reduction that translates inapproximability results to corresponding lower bounds on the price of anarchy. In the context of congestion games, he derived stronger inapproximability bounds for Rosenthal's congestion model involving polynomial cost functions with non-negative coefficients.

Computing a socially optimal profile has also been studied in the congestion model of Milchtaich [20], where resource costs are player-specific. Chakrabarty et al. [6] proved that the social optimum is inapproximable within any finite factor, unless $P = NP$. They exhibited some special cases in which a minimum cost solution can be found in polynomial time, e.g. when the number of strategies is bounded. Blumrosen and Dobzinski [5] considered the problem of maximizing welfare instead of minimizing costs and presented an 18-approximation for this problem. Assuming non-decreasing cost functions, they improved the approximation guarantee to $\frac{e}{e-1}$. De Keijzer and Schäfer [8] studied congestion games with positive externalities, where players benefit from other players choosing the same resource. They showed even very special cases of the problem are NP-hard and provided several approximation algorithms.

Computing equilibria. In the past decade the computational complexity to find a pure Nash equilibrium (PNE) in congestion games has been studied extensively. Ackermann et al. [1] proved that for congestion games with non-decreasing cost functions, matroids are the maximal property on the strategy space of every player that guarantees that best responses for players converge to a PNE in polynomial time. Fabrikant et al. [9] showed that a PNE can be found in polynomial time in symmetric network congestion games with non-decreasing cost functions. However, for general network games with non-decreasing cost functions, finding a PNE is PLS-complete [9]. These results have been strengthened to hold even when the cost functions are non-decreasing and linear [1]. It is also PLS-complete, for any $\alpha > 1$, to find α-approximate PNEs in congestion games [24], in which no player can unilaterally improve his cost by more than a factor α.

In singleton congestion games where all players have the same strategy space, the best Nash equilibrium can be found in polynomial time for any cost function [18]. Sperber [25] showed that both the best and the worst PNE can be found in polynomial time for non-decreasing cost functions with a greedy algorithm. However, she proved that in series–parallel graphs this is NP-hard, except in the case of finding the best PNE in a 2-player