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An average-case asymptotic analysis of the Container Relocation Problem

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ABSTRACT

The Container Relocation Problem (CRP) involves finding a sequence of moves of containers that minimizes the number of relocations needed to retrieve all containers in a given order. In this paper, we focus on average case analysis of the CRP when the number of columns grows asymptotically. We show that the expected minimum number of relocations converges to a simple and intuitive lower-bound for which we give an analytical formula.

1. Introduction

Due to limited space in the storage area in container terminals, containers are stacked in columns on top of each other. As shown in Fig. 1, several columns of containers create bays of containers. If a container that needs to be retrieved (*target* container) is not on the topmost tier of a column and is covered by other containers, the *blocking* containers must be relocated to other slots. As a result, during the retrieval process, one or more relocation moves are performed by the yard cranes. Finding the sequence of moves that minimizes the total number of relocations while retrieving containers from a bay in a pre-defined order is referred to as the Container Relocation Problem (CRP) or the Block Relocation Problem (BRP). For reviews and classification surveys of the existing literature on the CRP, we refer the reader to [3,6].

A common assumption of the CRP is that only the containers that are blocking the target container can be relocated. We refer to the CRP with this setting as the *restricted CRP*. In this paper, unless stated otherwise, CRP refers to the restricted CRP. On the other hand, if we relax this assumption, we will refer to the problem as *unrestricted CRP*.

In this paper, we study the CRP for large randomly distributed bays and we show that the ratio between the expected minimum number of relocations and a simple lower bound (developed by [2] and denoted below by S_0) approaches 1. While the problem is known to be NP-hard, this gives strong evidence that the CRP is "easier" to solve for large instances, and that heuristics can find near-optimal solutions.

Let us define the problem more formally: we are given a bay with C columns, P tiers; initially N containers are stored in the bay with exactly h containers in each column, where $h \leqslant P - 1$, so $N = h \times C$. We denote such a bay $B_{h,C}$. We label the containers based on their required departure order, i.e., container 1 is the first one to be retrieved. The CRP corresponds to finding a sequence of moves to retrieve containers $1, 2, \ldots, N$ (respecting the order) with a minimum number of relocations. For bay $B_{h,C}$, we denote the minimum number of relocations by $z_{opt}(B_{h,C})$. We focus on an average case analysis when the number of columns grows asymptotically. In our model, since $N = h \times C$, when C grows to infinity, N also grows to infinity.

Average case analysis of CRP is fairly new. The only other paper found in the literature is by [5]. They also provide a probabilistic analysis of the asymptotic CRP when both the number of columns and tiers grow to infinity. They show that there exists a polynomial time algorithm that solves this problem close to optimality with high probability. Our model departs from theirs in two aspects: (i) We keep the maximum height (number of tiers) a constant whereas in [5] the height also grows. Our assumption is motivated by the fact that the maximum height is limited by the crane

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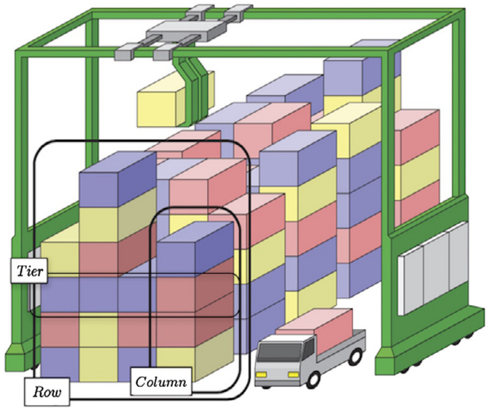

Fig. 1. Bays of containers in storage area.

6	5	3	10
4	2	7	11
1	8	9	12

Fig. 2. Bay with 4 tiers and 4 columns.

height, and it cannot grow arbitrarily; and (ii) We assume the ratio of the number of containers initially in the bay to the bay size (i.e., number of columns) stays constant (i.e., the bay is almost full at the beginning) and is equal to h. On the other hand, in [5], the ratio of the number of containers initially in the bay to the bay size decreases (and it approaches zero) as the number of columns grows. In other words, in the model of [5], in large bays, the bay is underutilized.

Before stating the main result in Section 3, we first provide four main ingredients in the next section: the notion of a uniformly random bay, the simple lower bound S_0 on the minimum number of relocations introduced by [2], a heuristic developed by [1] that performs well in large bays, and the notion of "special" columns.

2. Background

2.1. Uniformly random bay

We view a bay as an array of $P \times C$ slots (see Fig. 2). The slots are numbered from bottom to top, and left to right from 1 to $P \times C$. The goal is to generate a bay $B_{h,C}$ with uniform probability, meaning each container is equally likely to be anywhere in the configuration, with the restriction that there are h containers per column. We first generate a uniformly random permutation of $\{1, \ldots, h \times C\}$ called π. Then we assign a slot for each container with the following relation: $B_{h,C}(i,j) = \pi(h \times (j-1) + i)$ for $i \leqslant h$ and $B_{h,C}(i,j) = 0$ for $i \geqslant h+1$. One can see that each bay is generated with probability $1/N!$. There is a one to one mapping between configurations with C columns and permutations of $\{1, \ldots, h \times C\}$, denoted by $\mathcal{S}_{h\times C}$. Finally, we denote the expectation of random variable X over this uniform distribution by $\mathbb{E}_{h,C}[X]$.

2.2. The counting lower bound $\mathbf{S_0}$

This bound was introduced in [2] and it is based on the following simple observation. In the initial configuration, if a container is blocking, then it must be relocated at least once. Thus we count the number of blocking containers in $B_{h,C}$, we denote it as $S_0(B_{h,C})$, and we have $z_{opt}(B_{h,C}) \geqslant S_0(B_{h,C})$. Note that if a container blocks more than one container, it is counted only once. In Lemma 1 we give an explicit formula for the expectation of S_0 under the uniform distribution.

Lemma 1. *Let $C, h \in \mathbb{N}$ and S_0 be the counting lower bound defined above, we have*

$$\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right] = \alpha_h \times C, \quad (1)$$

where $\alpha_h = h - \sum_{i=1}^{h} 1/i$ is the expected number of blocking containers in one column.

Fact 2. *Note that α_h only depends on h.*

Proof of Lemma 1. Let $S_0^i(B_{h,C})$ be the number of blocking containers in column i. By the linearity of expectation, we have $\mathbb{E}_{h,C}\left[S_0(B_{h,C})\right] = \mathbb{E}_{h,C}\left[\sum_{i=1}^{C} S_0^i(B_{h,C})\right] = \sum_{i=1}^{C} \mathbb{E}_{h,C}\left[S_0^i(B_{h,C})\right] = \alpha_h \times C$, where $\alpha_h = \mathbb{E}_{h,C}\left[S_0^1(B_{h,C})\right] = \mathbb{E}_{h,1}\left[S_0(B_{h,1})\right]$. This relies on the fact that each column is identically distributed.

Now let us compute α_h. It is clear that $\alpha_1 = 0$. For $h \geqslant 2$, by conditioning on the event that the topmost container is the smallest number in the column or not, we obtain the recursive equation $\alpha_h = \alpha_{h-1} + (h-1)/h$. Finally by induction we have $\alpha_h = h - \sum_{i=1}^{h} 1/i$ which completes the proof.

2.3. The heuristic H ([1])

Suppose n is the target container located in column c, and r is the topmost blocking container in c. For convenience, we denote by $\min(c_i)$ the minimum of column c_i (note that $\min(c_i) = N+1$ if c_i is empty). H uses the following rule to determine $c^* \neq c$, the column where r should be relocated to. If there is a column c_i with $|c_i| < P$, where $\min(c_i)$ is greater than r, then H chooses such a column where $\min(c_i)$ is minimized, since columns with larger minimums can be useful for larger blocking containers (as r will never be relocated again, we say this relocation of r is a "good" move). If there is no column satisfying $\min(c_i) > r$ (any relocation of r can only result in a "bad" move), then H chooses the column where $\min(c_i)$ is maximized in order to delay the next unavoidable relocation of r as much as possible. We will refer to this heuristic as heuristic H and denote its number of relocations by $z_H(B_{h,C})$. Notice that $z_{opt}(B_{h,C}) \leqslant z_H(B_{h,C})$. Finally we state the following simple fact, which does not require a formal proof:

Fact 3. *For any configuration B with C columns and at most C containers, we have*

$$S_0(B) = z_{opt}(B) = z_H(B). \quad (2)$$

2.4. Definition of "special" columns

For $h, C \in \mathbb{N}$, a column in $B_{h,C}$ is called "special" if all of its containers belong to the C highest. Given this definition, a column in $B_{h,C+1}$ is "special" if all containers belong to the $C+1$ highest, or equivalently, if each of its containers has index at least $\omega_{h,C} = (h-1)(C+1)+1$. We will also consider the following event:

$$\Omega_{h,C} = \left\{B_{h,C+1} \text{ has at least 1 "special" column}\right\}. \quad (3)$$

Lemma 4 states that the event $\Omega_{h,C}$ has a probability that increases exponentially fast to 1 as a function of C. The proof of Lemma 4 can be found in the Appendix.

Lemma 4. *Let $h, C \in \mathbb{N}$ such that $C \geqslant h+1$ and $\Omega_{h,C}$ be the event defined by Eq. (3), then we have*

$$\mathbb{P}(\overline{\Omega_{h,C}}) \leqslant e^{-\theta_h (C+1)}, \quad (4)$$

where

$$\theta_h = \frac{1}{8h}\left(\frac{2}{h(h+1)}\right)^{2h} > 0. \quad (5)$$