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Continuous-time Markowitz's model with constraints on wealth and portfolio

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ABSTRACT

We consider a continuous-time Markowitz's model with bankruptcy prohibition and convex cone portfolio constraints. We first transform the problem into an equivalent one with bankruptcy prohibition but *without* portfolio constraints. The latter is then treated by martingale theory. This approach allows one to directly present the semi-analytical expressions of the pre-committed efficient policy without using the viscosity solution technique but within the framework of cone portfolio constraints. The numerical simulation also sheds light on results established in this paper.

1. Introduction

Since Markowitz [13] published his seminal work on the mean–variance portfolio selection, the mean-risk portfolio selection framework has become one of the most prominent topics in quantitative finance. Recently, there has been increasing interest in studying the dynamic mean–variance portfolio problem with various constraints. Typical contributions include [1–12,14]. The dynamic mean–variance problem can be treated in a forward-looking way by starting with the initial state. In some financial engineering problems, however, one needs to study stochastic systems with constrained conditions, such as cone-constrained policies. This naturally results in a continuous-time mean–variance portfolio selection problem with *constraints for the wealth process* (see [1]), and/or *constraints for the policies* (see [9,12]). To the best of our knowledge, despite active research efforts put in this direction in recent years, there has barely any progress in the study of the continuous-time mean–variance problem with the *mixed* restriction of bankruptcy prohibition and convex cone portfolio constraints. In this paper, we aim to address this long-standing and notoriously difficult problem, not only for its theoretical significance, but also for its practical importance. Our new approach, significantly different from those developed in the existing literature, hopes to establish a general theory for stochastic control problems with mixed constraints for both state and control variables.

The existing theories and methods cannot easily handle the mean–variance problem with the mixed restriction of bankruptcy prohibition and convex cone portfolio constraints. We find out that the market price of risk in policy is actually independent of the wealth process. This important finding allows us to overcome the difficulty of the original problem. In fact, we first show that the problem with the mixed restriction is equivalent to one only with bankruptcy prohibition via studying the Hamilton–Jacobi–Bellman (HJB) equations of the two problems. We then discuss the equivalent problem using the results obtained in [1].

2. Problem formulation and preliminaries

2.1. Notation

We use the following notation throughout the paper:

M': the transpose of any matrix or vector M;
$|a|$: $=\sqrt{\sum_i a_i^2}$ for any vector $a=(a_i)$;
$\|M\|$: $=\sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix $M=(m_{ij})$;
$\mathbb{R}^m$: m dimensional real Euclidean space;
$\mathbb{R}^m_+$: the subset of $\mathbb{R}^m$ consisting of elements with nonnegative components;

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$\mathbf{1}_A$: the indicator function for an event A that is equal to 1 if A happens, and 0 otherwise.

The underlying uncertainty is generated on a fixed filtered complete probability space $(\Omega, \mathbf{F}, \mathbf{P}, \{\mathcal{F}_t\}_{t\geqslant 0})$ on which is defined a standard $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted m-dimensional Brownian motion $W(\cdot) \equiv (W^1(\cdot), \dots, W^m(\cdot))'$. Given a Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_{\mathcal{H}}$, we can define a Banach space

$$L^2_{\mathcal{F}}(a, b; \mathcal{H}) = \left\{ \varphi(\cdot) \middle| \begin{array}{l} \varphi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable} \\ \text{process on } [a, b] \text{ and } \|\varphi(\cdot)\|_{\mathcal{F}} < +\infty \end{array} \right\}$$

with the norm $\|\varphi(\cdot)\|_{\mathcal{F}} = \left(\mathbf{E}\left[\int_a^b \|\varphi(t,\omega)\|^2_{\mathcal{H}} \mathrm{d}t\right]\right)^{\frac{1}{2}}$.

2.2. Problem formulation

Consider an arbitrage-free financial market where $m+1$ assets are traded continuously on a finite horizon $[0, T]$. One asset is a *bond*, whose price $S_0(t)$ evolves according to the ordinary differential equation

$$\begin{cases} \mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t, & t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)$ is the interest rate of the bond at time t. The remaining m assets are *stocks*, and their prices are modeled by the system of stochastic differential equations

$$\begin{cases} \mathrm{d}S_i(t) = S_i(t)\left\{ b_i(t)\mathrm{d}t + \sum_{j=1}^m \sigma_{ij}(t)\mathrm{d}W^j(t) \right\}, & t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases}$$

where $b_i(t)$ is the appreciation rate of the ith stock and $\sigma_{ij}(t)$ is the volatility coefficient at time t. Denote $b(t) := (b_1(t), \dots, b_m(t))'$ and $\sigma(t) := (\sigma_{ij}(t))$. We assume throughout that $r(t)$, $b(t)$ and $\sigma(t)$ are given deterministic, measurable, and uniformly bounded functions on $[0, T]$. In addition, we assume that the non-degeneracy condition on $\sigma(\cdot)$, that is,

$$y'\sigma(t)\sigma(t)'y \geqslant \delta y'y, \quad \forall\, (t, y) \in [0, T] \times \mathbb{R}^m, \tag{1}$$

is satisfied for some scalar $\delta > 0$. Also, we define the excess return vector $B(t) = (b_1(t) - r(t), \dots, b_m(t) - r(t))$.

Suppose an agent has an initial wealth $x_0 > 0$ and the total wealth of his position at time t is $X(t)$. Denote by $\pi_i(t)$, $i = 1, \dots, m$, the total market value of the agent's wealth in the ith stock at time t. We call $\pi(\cdot) := (\pi_1(\cdot), \dots, \pi_m(\cdot))' \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ a portfolio. We will consider self-financing portfolios here. Then it is well-known that $X(\cdot)$ follows (see [14])

$$\begin{cases} \mathrm{d}X(t) = [r(t)X(t) + \pi(t)'B(t)]\mathrm{d}t + \pi(t)'\sigma(t)\mathrm{d}W(t), \\ X(0) = x_0. \end{cases} \tag{2}$$

An important restriction considered in this paper is the convex cone portfolio constraints, that is $\pi(\cdot) \in \mathcal{C}$, where

$$\mathcal{C} = \left\{ \pi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \colon C(t)'\pi(t) \in \mathbb{R}^k_+, \forall\, t \in [0, T] \right\},$$

and $C : [0, T] \mapsto \mathbb{R}^{m\times k}$ is a given deterministic and measurable function. Another important restriction considered in this paper is the prohibition of bankruptcy, namely

$$X(t) \geqslant 0, \quad \forall\, t \in [0, T]. \tag{3}$$

Meanwhile, borrowing from the money market (at the interest rate $r(\cdot)$) is still allowed; that is, the money invested in the bond $\pi_0(\cdot) = X(\cdot) - \sum_{i=1}^m \pi_i(\cdot)$ has no constraint.

Definition 1. A portfolio $\pi(\cdot)$ is called an admissible control (or portfolio) if $\pi(\cdot) \in \mathcal{C}$ and the corresponding wealth process $X(\cdot)$ defined in (2) satisfies (3). In this case, the process $X(\cdot)$ is called an admissible wealth process, and $(X(\cdot), \pi(\cdot))$ is called an admissible pair.

Remark 1. In view of the boundedness of $\sigma(\cdot)$ and the non-degeneracy condition (1), we have that $\pi(\cdot) \in L^2_{\mathcal{F}}(a, b; \mathbb{R}^m)$ if and only if $\pi'(\cdot)\sigma(\cdot) \in L^2_{\mathcal{F}}(a, b; \mathbb{R}^m)$. The latter is often used to define the admissible process in the literature, for instance, [1].

Remark 2. It is easy to show that the set $\{\mathbf{E}[X(T)] : X(\cdot)$ is an admissible process$\}$ is an interval.

Mean–variance portfolio selection refers to the problem of, given a favorable mean level d, finding an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints), such that the expected terminal wealth $\mathbf{E}[X(T)]$ is d while the risk measured by the variance of the terminal wealth

$$\mathbf{Var}(X(T)) = \mathbf{E}[X(T) - \mathbf{E}[X(T)]]^2 = \mathbf{E}[X(T) - d]^2$$

is minimized. The following assumption is standard in the mean–variance literature (see, e.g., Assumption 2.1 in [12]).

Assumption 1. The value of the expected terminal wealth d satisfies $d \geqslant x_0 e^{\int_0^T r(s)\mathrm{d}s}$.

Definition 2. The mean–variance portfolio selection problem is formulated as the following optimization problem parameterized by d:

$$\begin{array}{ll} \min\limits_{\pi(\cdot)} & \mathbf{Var}(X(T)) = \mathbf{E}[X(T) - d]^2, \\ \text{subject to} & \begin{cases} \mathbf{E}[X(T)] = d, \\ \pi(\cdot) \in \mathcal{C},\ X(\cdot) \geqslant 0, \text{ and} \\ (X(\cdot), \pi(\cdot)) \text{ satisfies the Eq. (2).} \end{cases} \end{array} \tag{4}$$

An optimal control satisfying (4) is called an efficient strategy, and $(\sqrt{\mathbf{Var}(X(T))}, d)$, where $\mathbf{Var}(X(T))$ is the optimal value of (4) corresponding to d, is called an efficient point. The set of all efficient points, when the parameter d runs over all possible values, is called the efficient frontier.

In the current setting, the admissible controls belong to a convex cone, so the value of the expected terminal wealth may not be arbitrary. Denote by $V(d)$ the optimal value of the problem (4). Denote

$$\widehat{d} = \sup\left\{ \mathbf{E}[X(T)] : X(\cdot) \text{ is an admissible process} \right\}.$$

Taking $\pi(\cdot) \equiv 0$, we see that $X(t) \equiv x_0 e^{\int_0^t r(s)\mathrm{d}s}$ is an admissible process, so $\widehat{d} \geqslant \mathbf{E}[X(T)] = x_0 e^{\int_0^T r(s)\mathrm{d}s}$. The following nontrivial example shows that it is possible that $\widehat{d} = x_0 e^{\int_0^T r(s)\mathrm{d}s}$.

Example 1. Let $B(\cdot) = -C(\cdot)\chi$, where χ is any positive vector of appropriate dimension. Then for any admissible control $\pi(\cdot) \in \mathcal{C}$, we have $\pi(\cdot)'B(\cdot) = -\pi(\cdot)'C(\cdot)\chi \leqslant 0$. Therefore, by (2),

$$\mathrm{d}\,(\mathbf{E}[X(t)]) = (r(t)\mathbf{E}[X(t)] + \mathbf{E}[\pi(t)'B(t)])\mathrm{d}t \leqslant r(t)\mathbf{E}[X(t)]\mathrm{d}t,$$

which implies $\mathbf{E}[X(T)] \leqslant x_0 e^{\int_0^T r(s)\mathrm{d}s}$. Hence $\widehat{d} = x_0 e^{\int_0^T r(s)\mathrm{d}s}$.

Theorem 1. *Assume that $\widehat{d} = x_0 e^{\int_0^T r(s)\mathrm{d}s}$. Then the optimal value of the problem (4) is 0.*

Proof. From Assumption 1 and with $\widehat{d} = x_0 e^{\int_0^T r(s)\mathrm{d}s}$, we obtain that the only possible value of d is $x_0 e^{\int_0^T r(s)\mathrm{d}s}$. Note that $(X(t), \pi(t)) \equiv (x_0 e^{\int_0^t r(s)\mathrm{d}s}, 0)$ is an admissible pair satisfying the constraint of the problem (4), so $V(d) \leqslant \mathbf{E}[X(T) - d]^2 = \mathbf{E}[x_0 e^{\int_0^T r(s)\mathrm{d}s} - d]^2 = 0$. The claim follows immediately. □