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A full-Newton step interior-point algorithm for linear optimization based on a finite barrier

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ABSTRACT

In this paper, we propose a finite barrier kernel function for primal–dual interior-point algorithm in linear optimization with a full-Newton step. To our best knowledge, it is the first time that the property of exponential convexity is used for full-Newton step interior-point methods(IPMs). Moreover, the analysis is simplified and the complexity of the algorithm coincides with the currently best iteration bound for linear optimization problems.

1. Introduction

Since the path-breaking paper of Karmarkar [6], linear optimization(LO) became an active area of research. Primal–dual methods have attracted a lot of interest. These methods involve Newton direction, which is closely related to the well-known primal–dual logarithmic barrier function. Peng et al. [10–12] replaced the logarithmic barrier by a so-called self-regular barrier function, and modified the search direction accordingly, then they obtained a large-update method for which the theoretical iteration bound is $O(\sqrt{n}(\log n)\log(n/\varepsilon))$. Bai et al. [1] introduced a new barrier kernel function which is not self-regular. Based on the finite kernel function they devised a new large-update method with the same iteration bound. Later, Wang et al. [14] and Bai et al. [2,3] extended the new efficient large-update primal–dual IPM for LO to semidefinite optimization (SDO), $P^*(\kappa)$-linear complementarity problem ($P^*(\kappa)$-LCP) and second-order cone optimization(SOCO). Cai et al. [4] extended the primal–dual IPM for LO in Bai et al. [1] to convex quadratic optimization (CQO) by simple complexity analysis.

Earlier, Darvay [5] proposed a new technique for finding a class of search directions. By using an algebraic equivalent transformation of the nonlinear equations which define the central path, the author designed a full-Newton step primal–dual path-following interior-point algorithm for LO with iteration bound $O(\sqrt{n}\log(n/\varepsilon))$. Recently, Zhang and Xu [15] presented a feasible IPM for solving the LO problems with reformulation of the central path. Using the same reformulation of the central path, Kheirfam [7] proposed a new primal–dual path-following interior-point algorithm for LCP over symmetric cones. Later on, Mansouri et al. [9] presented a modified infeasible interior-point algorithm for LO.

In this paper, we introduce a full-Newton step interior-point algorithm for LO by modifying the Newton direction based on a finite barrier kernel function. We modify the feasibility step and use the property of exponential convexity to analyze the algorithm. We will adopt the basic analysis used in [1], and revise them to be suited for the small-update case. It is proved, for the feasible case, that the complexity obtained coincides with the best known bound, namely, $O(\sqrt{n}L)$, where L denotes the input data length from simple analysis.

The paper is organized as follows. In Sections 2 and 3, we review and develop some useful properties of the finite kernel function that are needed in the analysis of the algorithm. Then the analysis and complexity bound of the algorithm are presented in Section 4. Finally, conclusions and remarks are given in Section 5.

Let us introduce some notations first. Let e be the vector with all entries 1, and $x = (x_1, \ldots, x_i, \ldots, x_n) \in \mathbb{R}^n$, where x_i is the ith component. Let $x_{min} = min\{x_i\}, 1 \le i \le n$ and X be the $n \times n$ diagonal matrix with x_i the diagonal entries. For any two vectors x and s, xs denotes the componentwise (or Hadamard)

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product of the two vectors. We also use the notation $xs^{-1} := x/s = [x_1/s_1, x_2/s_2, \ldots, x_n/s_n]$, for $x, s \in \mathbb{R}^n$ such that $s_i \neq 0, i = 1, 2, \ldots, n$. We use $\|\cdot\|$ to denote the Euclidean norm $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the infinity norm.

2. Preliminaries

In this paper, we deal with primal–dual IPMs for solving the following LO problem:

$$(P)\ \min c^T x, \quad s.t.\, Ax = b,\ x \geq 0, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ satisfies $rank(A) = m, c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and its dual problem

$$(D)\ \max b^T y, \quad s.t.\, A^T y + s = c,\ s \geq 0. \tag{2}$$

It is assumed in IPMs theory that both (P) and (D) satisfy the interior point condition (IPC), i.e., there exists an (x^0, y^0, s^0) such that

$$Ax^0 = b, \quad x^0 > 0, \qquad A^T y^0 + s^0 = c, \quad s^0 > 0. \tag{3}$$

Finding optimal solutions of (P) and (D) is equivalent to solving the following system:

$$Ax = b, \quad x \geq 0, \qquad A^T y + s = c, \quad s \geq 0,\ xs = 0. \tag{4}$$

The basic idea of primal–dual IPMs is to replace the complementarity condition $xs = 0$ by the parameterized equation $xs = \mu e$. The replacement will give us the following new system:

$$Ax = b, \quad x \geq 0, \qquad A^T y + s = c, \quad s \geq 0,\ xs = \mu e. \tag{5}$$

If the IPC holds, then for each $\mu > 0$, system (5) has a unique solution. This solution, denoted by $(x(\mu), y(\mu), s(\mu))$, is called the μ-center of the primal–dual pair (P) and (D). The set of μ-centers with all $\mu > 0$ gives the central path of (P) and (D). It has been shown that the limit of the central path (as μ goes to zero) exists and it is an optimal solution of (P) and (D) (see Roos et al. [13]).

For a given feasible point (x, y, s), applying Newton's method to (5) gives the following linear system of equations

$$A\triangle x = 0, \qquad A^T \triangle y + \triangle s = 0, \quad s\triangle x + x\triangle s = \mu e - xs, \tag{6}$$

where $(\triangle x, \triangle y, \triangle s)$ gives the Newton step.

Define the vector

$$v := \sqrt{xs/\mu}. \tag{7}$$

Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$.

We use the notations

$$\bar{A} := AV^{-1}X/\mu = AS^{-1}V \tag{8}$$

and define the scaled search directions d_x and d_s by

$$d_x := v\Delta x/x, \qquad d_y := \Delta y, \qquad d_s := v\Delta s/s. \tag{9}$$

Then system (6) becomes

$$\bar{A}d_x = 0, \qquad \bar{A}^T d_y + d_s = 0, \quad d_x + d_s = v^{-1} - v. \tag{10}$$

The third equation in the above system is called the scaled centering equation. Let $\Psi(v) = \sum_{i=1}^n \psi(v_i)$ and $\psi(t) = (t^2 - 1)/2 - \log t, t > 0$, where ψ is called a kernel function. It can be easily verified

$$d_x + d_s = v^{-1} - v = -\nabla\Psi(v). \tag{11}$$

We use the norm-based proximity measure $\delta(v)$ defined by

$$\delta(v) := \|\nabla\Psi(v)\| = \sqrt{\sum_{i=1}^n (\psi'(t))^2}. \tag{12}$$

Note that since $\Psi(v)$ is strictly convex and minimal at $v = e$, so the minimal value is zero, we have

$$\Psi(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e. \tag{13}$$

Since d_x and d_s are orthogonal, we will have $d_x = 0$ and $d_s = 0$ if and only if $v = e$, i.e.,

$$d_x = 0 \text{ and } d_s = 0 \Leftrightarrow x = x(\mu) \text{ and } s = s(\mu).$$

The generic form of the algorithm is shown as follows:

Algorithm 1 Generic feasible IPM for LO

Input:
A threshold parameters $\tau > 0$;
An accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta, 0 < \theta < 1$;
feasible pair (x^0, y^0, s^0) with $\mu^0 > 0$, $v^0 = \sqrt{x^0 s^0/\mu^0}$ such that $\delta(v^0) \leq \tau$.
begin:
$x := x^0, y := y^0, s := s^0, \mu := \mu^0$;
while $n\mu \geq \varepsilon$ **do**
  $x := x + \Delta x$;
  $s := s + \Delta s$;
  $y := y + \Delta y$;
  $\mu := (1 - \theta)\mu$;
end while
end

3. Properties of the new proximity function

The aim of this paper is to investigate the kernel function, which has a finite value at the boundary of feasible region, namely

$$\psi(t) = (t - 1)^2/2. \tag{14}$$

Note that the kernel function can be considered as a special case of the kernel function of [1]

$$\psi(t) = (t^2 - 1)/2 + e^{\sigma(1-t)-1}/\sigma. \tag{15}$$

But there is a slight difference. In [1], it is required that all the components of vector v are greater than or equal to some value so that σ can be large enough, but in our paper, $\sigma \to 0$. The kernel function in (14) is first used in Liu and Sun [8].

Obliviously, $\psi'(t) = t - 1, \psi''(t) = 1, \psi'''(t) = 0$. The scaled centering equation becomes

$$d_x + d_s = p, \tag{16}$$

where $p = -\nabla\Psi(v) = e - v$.

The search direction decided by the kernel function coincides with the equivalent reformulation of the central path in Zhang and Xu [15], and is only up to a constant comparing with the equivalent algebraic transformation $\varphi(t) = \sqrt{t}$ proposed by Darvay [5].

Lemma 3.1. *One has* $\psi(t) = \psi'(t)^2/2, \Psi(v) = \delta(v)^2/2$ *and* $\delta(v) = \|p\|$.

Lemma 3.2. *Let* $t_1 \geq 1/2$ *and* $t_2 \geq 1/2$. *Then*

$$\psi(\sqrt{t_1 t_2}) \leq (\psi(t_1) + \psi(t_2))/2.$$