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Demand-flow of agents with gross-substitute valuations

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ABSTRACT

We consider the gross-substitute (GS) condition introduced by Kelso and Crawford (1982). GS is a condition on the demand-flow in a specific scenario: some items become more expensive while other items retain their price. We prove that GS is equivalent to a much stronger condition, describing the demand-flow in the general scenario in which all prices may change: the demand of GS agents always flows (weakly) downwards, i.e., from items with higher price-increase to items with lower price-increase.

1. Introduction

Many markets involve a set of distinct indivisible goods that can be bought and sold for money. The analysis of such markets crucially depends on the agents' *valuation functions*—the functions that assign monetary values to bundles. It is common to assume that agents' valuations are weakly increasing (more goods mean weakly more value) and quasi-linear in money. Even so, without further restrictions on the valuations, the market may fail to have desirable properties such as the existence of a price-equilibrium. Kelso and Crawford [4] introduced a property of valuations which they called *gross-substitutes (GS)*. An agent's valuation has the GS property if, when the prices of some items increase, the agent does not decrease its demand for the other items. Kelso and Crawford [4] proved that a market in which all agents are GS always has a price-equilibrium. Gul and Stacchetti [2,3] complemented this result by proving that the GS condition is, in some sense, necessary to ensure existence of a price-equilibrium. The GS condition has been widely used in the study of matching markets [8], auctions [5] and algorithmic mechanism design [6].

The GS condition specifies the behavior of an agent in a very specific situation: some items become more expensive, while other items retain their original price. In this paper we characterize the behavior of GS agents in the more general situation, in which the prices of all items change in different ways and in different directions. This characterization may have several potential applications:

(a) Analyzing the response of markets to exogenous shocks. For example, suppose the government puts price-ceilings on several items. With a single item-type, it is obvious that a price-ceiling below the equilibrium-price will result in excess demand. But with multiple item-types, this is not necessarily so. For example, it is possible that the prices of both item x and item y are below their equilibrium prices, but because of substitution effects, buyers switch from demanding y to demanding x so the net effect is an excess supply in y and an excess demand in x. In order to analyze such markets, we have to understand how exactly agents move from one item-type to another when the prices change.

(b) Designing dynamic combinatorial auctions. In such an auction, the auctioneer modifies the prices of different items at different rates in an attempt to change the aggregate demand. Gul and Stacchetti [3] describe one such auction, in which the prices are always ascending. In order to design different auctions, it may be useful to know the effect of different price-changes on the agents' demand.

(c) Using field-data to detect the existence of complementarities (i.e., valuations that are *not* GS) by comparing demands under different price-vectors.

(d) Our original application [9] was a double-auction mechanism where the market-prices are set by the auctioneer in a way that guarantees truthfulness but might not be entirely efficient; understanding the demand-flow of agents lets us calculate an upper bound on the loss of efficiency.

Consider two price-vectors: old and new. For every item x, define Δ_x as the price-increase of x (the new price minus the old price). Add a "null item" $\emptyset$ and set its price-increase to 0. Arrange the items vertically by ascending price-increase. Then, our main result is that:

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The demand of a gross-substitute agent always flows weakly downwards.

I.e., an agent may switch from wanting an item whose price increased more to an item whose price increased less, but not vice versa. This property is trivially true for a unit-demand agent, but it is not true when the agent regards some items as complementaries.

Example 1.1. There are three items: x, y, z. Initially their prices are \$10, \$10, \$10. Then, the prices increase by $\Delta_x = \$20$, $\Delta_y = \$30$, $\Delta_z = \$40$, so that the new prices are \$30, \$40, \$50. Consider two agents with the following valuations:

	x	y	z	$x+y$	$x+z$	$y+z$	$x+y+z$
Alice	\$65	\$70	\$75	\$70	\$75	\$75	\$75
Bob	\$40	\$40	\$66	\$80	\$75	\$75	\$80

Alice has unit-demand: she needs only one item and values each bundle as the maximum item in that bundle. Bob regards x and y as complementaries: each of them alone is worth less than z, but together they are worth more than $x+z$ and $y+z$. (Note that Bob's valuation is submodular but not GS.)

In the initial prices Alice's preferred bundle is z, and after the price-change her preferred bundle is x, so her demand flows downwards—toward the smaller price-increase.

In contrast, Bob's demand is initially $x + y$, and after the price-change his demand is z, so his demand flows upwards—toward the item with the larger price-increase. □

Our main result is that GS agents behave like unit-demand agents in this regard: their demand flows only downwards.

2. Model and notation

There is a finite set of indivisible items, $M = \{1, \ldots, m\}$. There is an m-sized price-vector p: a price per item. The price of a bundle is the sum of the prices of the items in it: $p(X) := \sum_{x\in X} p_x$.

The present paper focuses on a single agent with a single valuation-function $u : 2^M \to \mathbb{R}$. u is assumed to be weakly-increasing: if a bundle $X \subseteq Y$ then $u(X) \leq u(Y)$.

The agent's utility is quasi-linear in money. Given a utility function u and a price-vector p, the agent's net-utility function u_p is: $u_p(X) := u(X) - p(X)$.

Definition 2.1. Given a valuation function u and a price-vector p, we say that a bundle P is a *p-demand* if it is optimal for the agent to buy this bundle when the prices are p, i.e., the set P maximizes the net-utility function $u_p(\cdot)$ over all bundles of items: $\forall X : u_p(P) \geq u_p(X)$.

Definition 2.2. Given a valuation function u and a price-vector p, we say that an item x is *p-demanded* if there exists a p-demand P such that $P \ni x$.

Definition 2.3. Given an agent, an item x and a pair of price-vectors (p, q), we say that:

(a) The agent *abandoned* item x if x is p-demanded but not q-demanded.
(b) The agent *discovered* item x if x is q-demanded but not p-demanded.

Definition 2.4 ([4]). An agent's valuation function has the *gross-substitute (GS)* property if, for every pair of price-vectors (p, q) such that $\forall y : \Delta_y \geq 0$, if $\Delta_x = 0$ then the agent has not abandoned x.

Definition 2.5. A valuation has the *downward-demand-flow (DDF)* property if the following are true for every pair of price-vectors (p, q) (where $\Delta_x := q_x - p_x$):

(a) If $\Delta_x \leq 0$ and the agent abandoned x, then he discovered some y with $\Delta_y < \Delta_x$.
(b) If $\Delta_x \geq 0$ and the agent discovered x, then he abandoned some y with $\Delta_y > \Delta_x$.

DDF implies GS: part (a) of the DDF definition implies the GS definition. Our main result is the converse implication: GS implies DDF.

3. M♮-concavity

Our main technical tool is the following characterization of GS valuations [1]:

Definition 3.1. A valuation function u is *$M^\natural$-concave* if-and-only-if, for every two bundles X, Y and for every $X' \subseteq X \setminus Y$ with $|X'| = 1$ (i.e., X' is a singleton), there exists a subset $Y' \subseteq Y \setminus X$ with $|Y'| \leq 1$ (i.e., Y' is either empty or a singleton) such that:

$$u(X \setminus X' \cup Y') + u(Y \setminus Y' \cup X') \geq u(X) + u(Y).$$

Lemma 3.1 ([1]). *A valuation function u is $M^\natural$-concave if-and-only-if it is gross-substitute.*

Using the $M^\natural$-concavity characterization, it is easy to prove that GS is preserved in net-utility functions and marginal-valuation functions:

Lemma 3.2. *Let p be an arbitrary price vector. A valuation function u is $M^\natural$-concave if-and-only-if the net-utility function u_p is $M^\natural$-concave.*

Definition 3.2. Given a valuation u and a constant bundle Z, the *marginal valuation u_{Z+}* is a function that returns, for every bundle X that does not intersect Z, the additional value that an agent holding Z gains from having X:

$$u_{Z+}(X) := u(Z \cup X) - u(Z) \quad \text{for all } X \text{ with } X \cap Z = \emptyset.$$

Lemma 3.3 ([7]). *A valuation function u is GS if-and-only-if, for every bundle Z, the marginal-valuation function u_{Z+} is GS.*

4. Telescopic arrangement of maximizing bundles

By definition, an agent's demanded bundles are *maximizing-bundles*—bundles that maximize his net-utility over all 2^m possible bundles. In addition to the global maximizing-bundles, we can consider the maximizing-bundles in each size-group, i.e., the maximizing-bundles among the bundles with 1 item, with 2 items, etc. In this section we prove that, when the agents' valuation is $M^\natural$-concave, the maximizing-bundles in the different size-groups have a telescopic arrangement: each maximizing-bundle contains smaller maximizing-bundles and is contained in larger maximizing-bundles.

Definition 4.1. Given valuation u on m items and a number $i \in \{0, \ldots, m\}$, a bundle Z_i is called *i-maximizer of* u if it maximizes u among all bundles with i items. I.e., $|Z_i| = i$ and for every other bundle X_i with i items, $u(Z_i) \geq u(X_i)$.

Lemma 4.1. *For every $M^\natural$-concave valuation u on m items and two integers i, j such that $0 \leq i < j \leq m$:*

(a) *For every i-maximizer Z_i there is a j-maximizer Z'_j such that $Z'_j \supset Z_i$.*
(b) *For every j-maximizer Z_j there is an i-maximizer Z'_i such that $Z_j \supset Z'_i$.*