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Refuting a conjecture of Goemans on bounded degree spanning trees

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ABSTRACT

In 2006, Goemans presented an approximation algorithm for the minimum bounded degree spanning tree problem that constructs a tree with cost at most the optimal value of an LP relaxation but degree bound violations of up to two units per vertex. He conjectured that violations of at most one per vertex are attainable, providing a second conjecture that would make his approach achieve this guarantee. While the first conjecture was answered positively by Singh and Lau, we refute the second.

1. Introduction

The minimum bounded degree spanning tree (MBDST) problem is the following: Given a graph $G = (V, E)$, costs $c : E \to \mathbb{R}$ and degree bounds $d : V \to \mathbb{Z}_{>0}$, find a minimum cost spanning tree T of G such that for all $v \in V$, the degree of v in T is at most $d(v)$.

It is well known that for any $D \geq 2$, MBDST is NP-hard even if all vertices have the same degree bound $d(v) = D$. In particular, for $D = 2$, the problem asks to find a Hamiltonian path. Thus interest arose in finding approximation algorithms with various trade-offs between cost of the spanning tree and violation of the degree bounds. While for the unweighted problem, the best possible approximation with degree bound violations of only at most one unit was found by Fürer and Raghavachari [6] in 1994, the analogous result for the weighted case remained open until 2007.

After a series of papers [3,4,8,9,12,13] made progress on the approximation guarantee of the weighted problem, Goemans [7] presented the first algorithm that returned a spanning tree violating the degree constraints by at most an additive constant, namely $+2$, and of cost no more than the optimal spanning tree that does not violate any degree constraint. The gap to degree bound violations of at most one unit per vertex for the weighted case was closed by Singh and Lau using an iterative relaxation technique [15]. There are many constrained spanning tree problems that are closely related to MBDST, and also generalise it in several ways (see [1,2,5,10,11,16] and references therein).

In this work, we refute a conjecture of Goemans that would allow for improving his approach in [7] to degree bound violations of at most one unit per vertex. Even though Singh and Lau [15] already obtained an optimal approximation algorithm through different techniques, such an extension of Goemans' algorithm would have been interesting. In particular, Goemans' approach shows that any solution to the natural LP relaxation of MBDST is contained in a matroid intersection polytope, all of whose vertices correspond to spanning trees violating each degree constraint by at most $+2$. This, for example, provides an easy way to decompose any LP solution as a convex combination of spanning trees with violation at most $+2$. Goemans' conjecture being true would have automatically extended these structural results to constraint violation of at most $+1$, which is best possible. We start by a brief description of Goemans' approach before stating and refuting his conjecture.

Goemans' approach considers the natural LP relaxation of MBDST, given by

$$\begin{array}{lrcll} \min & c^\top x & & & \\ \text{s.t.} & x(E[S]) & \leqslant & |S|-1 & \forall\, S \subset V,\ S \neq \emptyset, \\ & x(E) & = & |V|-1, & \\ & x(\delta(v)) & \leqslant & d(v) & \forall\, v \in V, \\ & x & \in & \mathbb{R}^E_{\geqslant 0}. & \end{array} \qquad (\mathrm{LP}_{\mathrm{MBDST}})$$

Here, $E[S]$ denotes the set of edges inside S, $\delta(v)$ denotes the set of edges incident to vertex v and $x(U) = \sum_{e \in U} x(e)$ for $U \subseteq E$. Based on this, Goemans' algorithm proceeds as follows:

1. Obtain an optimal vertex solution x^* to ($\mathrm{LP}_{\mathrm{MBDST}}$), and let $E^* = \operatorname{supp}(x^*) = \{e \in E \mid x(e) > 0\}$.
2. Orient the edges of the graph (V, E^*) to obtain a directed graph (V, A^*) such that all indegrees are at most two.

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3. Using matroid intersection, compute a minimum spanning tree within E^*, i.e., a basis of the graphic matroid M_1 over E^*, that is simultaneously independent in the partition matroid M_2 whose independent sets are given by $\mathcal{I} = \{F \subseteq E^* \mid |F \cap \delta^+_{A^*}(v)| \leqslant d(v)\ \forall v \in V\}$.

Here, $\delta^+_{A^*}(v)$ denotes the set of edges going out of v in the oriented graph (V, A^*). The key result that Goemans proves for obtaining the orientation is that (V, E^*) is sparse in the sense that for all non-empty $U \subseteq V$, we have $|E^*[U]| \leqslant 2|U| - 3$. This property implies that an orientation as described in step 2 exists. Moreover, such an orientation can be found efficiently. The orientation guarantees that the tree returned by the matroid intersection violates the degree bounds by at most two units: At every vertex $v \in V$, at most $d(v)$ of the outgoing edges appear in T, and there are at most two additional incoming edges, which leads to a total of at most $d(v) + 2$ edges in T that may be incident with v.

Therefore, the crucial step in Goemans' approach is to get a good orientation of the graph (V, E^*) that allows for a relatively accurate description of the degree constraints by a matroid. This idea leads to the following conjecture on the existence of an even better orientation.

Conjecture 1 (*Goemans, [7]*)**.** *Let x^* be a vertex solution of* (LP$_{\text{MBDST}}$)*, and let $E^* = \operatorname{supp}(x^*)$. Then, there exists an orientation A^* of E^* such that for all $v \in V$, we have*

$$\sum_{e \in \delta^-_{A^*}(v)} \left(1 - x^*(e)\right) \leqslant 1.$$

As Goemans [7] showed, if this conjecture was true, the matroid M_2 could be replaced by a different partition matroid with independent sets $\mathcal{I} = \{F \subseteq E^* \mid |F \cap \delta^+_{A^*}(v)| \leqslant \lceil x^*(\delta^*_{A^*}(v)) \rceil\ \forall v \in V\}$, leading to a spanning tree that violates the degree bounds by at most one unit.

We are able to refute an even weaker version of Conjecture 1. Let $z^* := 1 - x^*$ denote the *spare vector*, then the conjecture states that for every edge e, the spare $z^*(e)$ can be assigned to one of the incident vertices such that in total, each vertex gets at most one unit of spare assigned to it. We show that even if we allow splitting the spare of every edge among the incident vertices, this is impossible. To do so, we construct an instance containing what we call an *obstruction*: A set $U \subseteq V$ such that $z^*(E[U]) > |U|$, i.e., there is a subgraph on $|U|$ vertices with more than $|U|$ units of spare on the induced edges. This clearly contradicts even the weakening of Conjecture 1.

The paper is organised as follows. In Section 2, we generalise the problem to rational degree bounds and construct a family of instances giving counterexamples of the prescribed type. Section 3 then explains how fractional degree bounds can be modelled in larger instances by attaching certain auxiliary graphs to vertices with fractional degree bounds.

For notational convenience, we define two polytopes associated to a graph $G = (V, E)$ and degree bounds $d : V \to \mathbb{Q}$, namely the *spanning tree polytope* $P_{\text{ST}}(G)$ and the *bounded degree spanning tree polytope* $P_{\text{BDST}}(G, d)$, given by

$$P_{\text{ST}}(G) := \left\{ x \in \mathbb{R}^E_{\geqslant 0} \middle| \begin{array}{c} x(E[S]) \leqslant |S| - 1\ \forall S \subset V,\ S \neq \emptyset, \\ x(E) = |V| - 1 \end{array} \right\},$$

$$P_{\text{BDST}}(G, d) := \left\{ x \in P_{\text{ST}}(G) \middle| x(\delta(v)) \leqslant d(v)\ \forall v \in V \right\}.$$

Note that while P_{ST} is integral for all G (see [14]), P_{BDST} is not, in general. Constraints of the form $x(E[S]) \leqslant |S| - 1$ and $x(\delta(v)) \leqslant d(v)$ are referred to as *spanning tree constraints* and *degree constraints*, respectively.

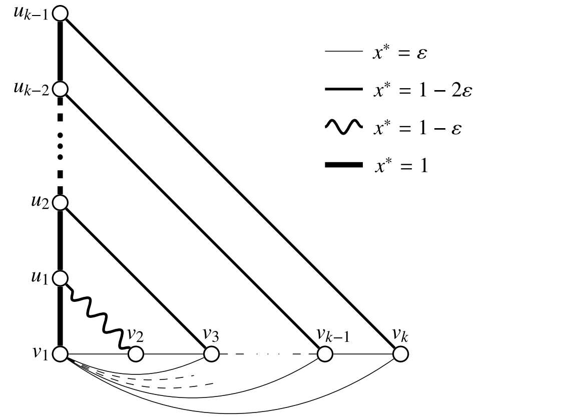

Fig. 1. The graph G and the point $x^* \in \mathbb{R}^E$.

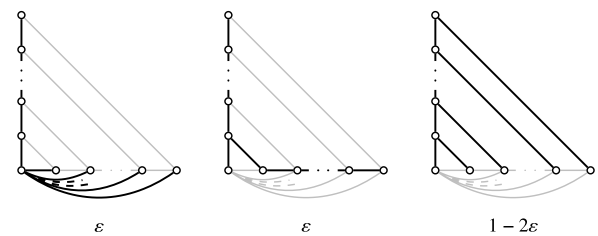

Fig. 2. Decomposition of x^* as convex combination of spanning trees.

2. Counterexamples with fractional degree bounds

For $k \in \mathbb{Z}_{>0}$ and $\varepsilon \in (0, 1/2) \cap \mathbb{Q}$, let the graph $G = (V, E)$ on the vertex set $V := \{u_1, \ldots, u_{k-1}\} \cup \{v_1, \ldots, v_k\}$ be as in Fig. 1, where we also indicate a point $x^* \in \mathbb{R}^E_{\geqslant 0}$ defined by

$$x^*(e) := \begin{cases} \varepsilon & \text{if } \exists\, i, j \in [k] : e = v_i v_j, \\ 1 - \varepsilon & \text{if } e = u_1 v_2, \\ 1 - 2\varepsilon & \text{if } \exists\, i \in \{2, \ldots, k-1\} : e = u_i v_{i+1}, \\ 1 & \text{if } e = v_1 u_1 \text{ or } \exists\, i \in [k-2] : e = u_i u_{i+1}. \end{cases}$$

Moreover, we define $d(v) := x^*(\delta(v))$ for all $v \in V$.

We now show that x^* is an extreme point of $P_{\text{BDST}}(G, d)$.

Lemma 2. *For $G = (V, E)$, $d : V \to \mathbb{Q}$ and $x^* \in \mathbb{R}^E_{\geqslant 0}$ as defined above, x^* is an extreme point of $P_{\text{BDST}}(G, d)$.*

Before proving the lemma, we observe that for any $k \geqslant 4$ and $\varepsilon < \frac{k-3}{2k-3}$, the extreme point x^* contains an obstruction, namely the set $U = \{v_1, \ldots, v_k\}$. Indeed, using the assumptions on k and ε, we see that the spare z^* satisfies

$$z^*(E[U]) = (2k - 3)(1 - \varepsilon) > k = |U|.$$

Hence, distributing spares in accordance with Conjecture 1 is impossible. To refute Goemans' conjecture, it suffices to show that we can "simulate" the fractional degree bounds d used to obtain the extreme point x^* with integral degree bounds (as a subsystem of a larger instance). This will be done in Section 3.

Proof of Lemma 2. We first show that $x^* \in P_{\text{BDST}}(G, d)$. The degree constraints $x(\delta(v)) \leqslant d(v)$ are satisfied at $x = x^*$ by definition of d for all $v \in V$, so it suffices to see $x^* \in P_{\text{ST}}(G)$. Equivalently, it suffices to prove that x^* can be written as a convex combination of indicator vectors of spanning trees of G. This can be achieved using the trees in Fig. 2 with the indicated coefficients.

Knowing $x^* \in P_{\text{BDST}}(G, d)$, it is enough to reveal a full-rank system of constraints that are tight at x^*. By definition of d, all degree constraints $x(\delta(v)) \leqslant d(v)$ are tight at x^*. However, we only use those for $v \in V \setminus \{v_1\}$. Among the spanning tree constraints