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Information, egalitarianism and the value

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ABSTRACT

This work proposes a value for a cooperative game with missing information. The value distributes the worth of the grand coalition on the basis of the number of known coalitions. When all the information is contained in the characteristic function the value coincides with the equal division solution. An axiomatic characterization of the value is presented which uses the nullifying player axiom introduced in van den Brink (2007).

1. Introduction

Despite it being over sixty years since it was first published, the most important single-valued solution for a cooperative game remains the Shapley value proposed in [11]. One important reason to explain this is that the Shapley value uses all the information contained in the characteristic function to determine each player's payoff. Also, the Shapley value admits a simple and convincing axiomatic characterization. Shapley himself expressed surprise at the characterization noting that "It is remarkable that no further conditions are required to determine the value uniquely". Another reason for the prominence of the Shapley value is the deep connections, explained in [1], between the Shapley value and the competitive equilibrium in smooth economies with an atomless continuum of individuals. However, because all the information about the worths of the coalitions is required to calculate the Shapley value, this is also a drawback of the solution. If the characteristic function represents a one-off interaction between the players it may be that there is no way of knowing the worths of particular coalitions. An alternative, which is easier to model using standard tools, is to consider the case where the characteristic function is uncertain. Many papers have suggested solutions to uncertain cooperative games, including [3,4,9,12]. Suppose, then, some information is missing from the characteristic function. How should an impartial observer distribute the worth of the grand coalition amongst the players? It is this question which this work tries to answer.

A cooperative game with missing information is a mapping from the power set of the players into the union of the real line and the empty set. If, for some coalition, their worth is equal to the empty set, then the worth of the coalition is treated as being missing from the game. Therefore, the model permits arbitrary information to be missing from the characteristic function. The only assumptions imposed upon the characteristic function are that the worth of the grand coalition is not equal to the empty set, and the worth of the empty set is zero. Ideally, one would like a value for a game to possess three key properties:

1. Be applicable to the whole domain under consideration.
2. Be easy to calculate—at least in games with small numbers of players.
3. It should have a clear axiomatic characterization which reveals the underlying behaviour of the value.

The solution advanced here satisfies these three properties. The value distributes the worth of the grand coalition between the players in proportion to the number of known coalitions of which each player is a member. It is such that when all the information is contained in the characteristic function it coincides with the equal division solution. Therefore, the value can be thought of as an extension of the equal division solution to a wider domain than is normally studied in the literature. An axiomatic characterization of the value is presented using four axioms: efficiency, additivity, nullifying player and known coalitions. Efficiency requires that the value always distributes the worth of the grand coalition amongst the players. The axiom of additivity states, in the usual way, that if the value is applied to the sum of games, then the payoff should be equal to the sum of the payoffs when the value is applied to the separate games. The axiom of nullifying player was used

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in [13] to highlight the differences between the Shapley value and egalitarian solutions. A player is a nullifying player if all the known coalitions which contain the player have a worth of zero. The axiom states that nullifying players receive zero payoff. Finally, the axiom of known information applies to non-nullifying players and requires that the payoffs be related to the number of known coalitions in the game. The reasoning behind the known coalitions axiom is that, if there is positive worth distributed amongst non-nullifying players, then the more known coalitions a player belongs to, the higher should be their payoff. When all the information is contained in the characteristic function, the axiom of known coalitions implies symmetric payoffs to non-nullifying players.

1.1. Related literature

The interest in axiomatic value theory began with [11] demonstrating that the axioms of efficiency, symmetry, additivity and carrier player characterize a unique solution on the space of cooperative games. Since that paper there has been sustained interest in axiomatic value theory. This brief survey only covers the most important, and recent contributions, to the literature. A succinct and detailed textbook treatment of value theory is contained in [7], especially chapters 18 and 20. In [10] the main alternative to the Shapley value was proposed, which is the nucleolus. The nucleolus lexicographically minimizes the ordered vector of excesses, and unlike the Shapley value, always selects from the core of a game whenever it is non-empty. Even if the core is empty, the nucleolus is often interpreted as the core-centre of a game. In a series of papers, [14,15] it was demonstrated that the mathematically convenient axiom of additivity could be dropped in the characterization of the Shapley value and replaced with the economically intuitive axiom of strong monotonicity (or marginalism).

In recent years, there has been particular interest in values which are more egalitarian than the Shapley value, or can be applied to wider domains. In [8] a solidarity value for a cooperative game was suggested. In this value, each player receives the average of the marginal contributions of the players preceding them in each permutation. By replacing the usual null player axiom with an average null player axiom, and finding a basis for a game which differed from the usual unanimity game basis, they were able to characterize the solidarity value. In [13] it was demonstrated that if the null player axiom is replaced with a nullifying player axiom in the usual Shapley characterization, then the axioms characterize the equal division solution (in which the worth of the grand coalition is divided equally between the players). Several other egalitarian solutions were also analysed, such as the centre of the imputation set, and an application to auction theory was presented. The work in [6] studied procedural values in which each player could receive particular shares of the marginal contributions of players preceding them in each permutation of the players. These procedural values include the Shapley value, the solidarity value and the equal division solution as special cases. In [5] the same model as in this paper was analysed, in which arbitrary information may be missing from a cooperative game, and considered three different ways in which the Shapley value may be extended to the missing information domain. In [2] a production economy was studied, and the axioms of efficiency, symmetry and monotonicity were shown to characterize overall proportional taxation rates.

2. The model

Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the finite set of players in the game. A **cooperative game with missing information** is (N, v) with v being the characteristic function $v : 2^N \to \Re \cup \{\varnothing\}$. A **coalition** is an $S \subseteq N$. The **grand coalition** is N. For any coalition $S \subseteq N$, if $v(S) \neq \varnothing$, then $v(S)$ is the money **worth** which S can obtain independently of the other players. If $v(S) = \varnothing$, then the worth of coalition S is missing from the game.

Assumption 2.1. For every $v : 2^N \to \Re \cup \{\varnothing\}$, $v(N) \neq \varnothing$ and $v(\varnothing) = 0$.

Let G denote the space of games satisfying Assumption 2.1. A **value** is a function $\varphi : G \to \Re^N$. For each $v \in G$ a value assigns a unique payoff vector in $\Re^N$. For any $v \in G$ let the set $K(v)$ be

$$K(v) = \{S \subseteq N : v(S) \neq \varnothing\}.$$

The set $K(v)$ is the set of **known coalitions** in the game v. Two games $v, w \in G$ will be called **comparable** if $K(v) = K(w)$. Let $K_i(v)$ be the set

$$K_i(v) = \{S \in K(v) : i \in S\}.$$

The set $K_i(v)$ contains those coalitions of which player i is a member and whose worths are known in the game v. Note that Assumption 2.1 ensures that for every $v \in G$, $|K_i(v)| \geq 1$. The following example should help to make these ideas clear.

2.1. An illustrative example

Suppose the set of players is $N = \{1, 2, 3\}$. The worth of the grand coalition is $v(N) = 8$. The worths of the smaller coalitions are

$$v(12) = v(13) = 8, \quad v(1) = 8, \quad v(\varnothing) = 0$$

and

$$v(S) = \varnothing \quad \text{for all other coalitions.}$$

How should the money worth of the grand coalition, 8, be distributed amongst the players? The payoff vector which this work proposes to the distribution problem is $(x_1, x_2, x_3) = (4, 2, 2)$. This distribution seems fair because there are more coalitions which contain player 1 whose worths are known. Also, there are the same number of known coalitions containing players 2 and 3, so it seems reasonable to give players 2 and 3 the same payoff.

2.2. The value and axioms

Here four axioms are defined: efficiency, additivity, nullifying player and known coalitions. The first is efficiency, which requires that the value always distributes all of the worth of the grand coalition amongst the players.

Definition 2.1. A value φ satisfies efficiency if $\sum_{i \in N} \varphi_i(N, v) = v(N)$ for every $v \in G$.

The axiom of additivity links the payoffs which players receive across the addition of different games. Before defining the axiom of additivity, let us define the sum of games on the missing information domain.

Definition 2.2. Suppose $v, w \in G$. If the games v and w are comparable, so $K(v) = K(w)$, the game $u = v + w$ is given by

$$u(S) = \begin{cases} v(S) + w(S) & \text{if } S \in K(v); \\ \varnothing & \text{if } S \notin K(v). \end{cases}$$

Addition of two games is only defined when the games have the same set of known coalitions.

Definition 2.3. A value φ satisfies additivity if, whenever $v, w \in G$ are comparable, $\varphi_i(N, v+w) = \varphi_i(N, v) + \varphi_i(N, w)$ for every $i \in N$.